Comments on the spectral concentration problem

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Motivation

The spectral concentration problem

H. J. Landau, H. O. Pollak, D. Slepian 1960s:

A function cannot be compactly supported in both space and Fourier domains. What if we look for functions that are compactly supported in one domain, and among all of them take the one that is the "most concentrated" in the other domain?

Heisenberg inequality: $f \in L^2(\mathbb{R}, \mathbb{C})$,

$$\|xf\|_{L^2}^2 \, \|\xi \widehat{f}\|_{L^2}^2 \geq \frac{\|f\|_{L^2}^4}{16\pi^2}$$

(equality iff Gaussians).

The spectral concentration problem

1960s formulation:

$$\max\left\{ \frac{\int_{-T}^{T} |f|^2}{\|f\|_{L^2}^2}, \quad \operatorname{supp} \hat{f} \subset [-W, W]. \right\}$$

$$f(x) = \frac{1}{2\pi} \int_{-W}^{W} \hat{f}(\xi) e^{i\xi x} \mathrm{d}\xi$$

$$\begin{aligned} \int_{-T}^{T} |f|^2 &= \frac{1}{4\pi^2} \int_{-W}^{W} \int_{-W}^{W} \int_{-T}^{T} \widehat{f}(\xi) \overline{\widehat{f}(\eta)} e^{ix\xi - ix\eta} \mathrm{d}\zeta \mathrm{d}\eta \mathrm{d}x \\ &= \frac{1}{4\pi^2} \int_{-W}^{W} \int_{-W}^{W} \widehat{f}(\xi) \overline{\widehat{f}(\eta)} \frac{\sin(T(\xi - \eta))}{\xi - \eta} \mathrm{d}\eta \end{aligned}$$

The spectral concentration problem

Eigenvalue problem:

$$(K\psi)(\xi) = \int_{-W}^{W} \frac{\sin(T(\xi-\eta))}{\xi-\eta}\psi(\eta)\mathrm{d}\eta = \lambda\psi(\xi), \quad \xi \in [-W, W].$$

Solution by Slepian *et al.*:

$$H = -\partial_\xi(W-\xi^2)\partial_\xi + T\xi^2 \quad \Longrightarrow \quad [H, K] = 0.$$

Eigenvalues of H: Prolate Spheroidal wave functions.

$$H\psi_j = \omega_j\psi_j, \quad \omega_{j+1} > \omega_j \quad \Longrightarrow \quad K\psi_j = \lambda_j\psi_j.$$

D. Slepian (1983): *There was a lot of serendipity here, clearly. And then our solution too, seemed to hinge on a lucky accident*

The spectral concentration problem

$$H = \partial_\xi (W^2 - \xi^2)\partial_\xi - T^2\xi^2 \quad \Longrightarrow \quad [H, K] = 0.$$

Fourier transform: exchange W and T.
After scaling:

$$H = -\partial_x (1 - x^2)\partial_x + c^2 x^2, \quad c = WT.$$

Result by Landau and Widom 1980. $K\psi_j = \lambda\psi_j$,

$$\begin{aligned} N(K, \alpha) &= \sharp \{ j \in \mathbb{N}, \quad \lambda_j \geq \alpha \} \\ &= \frac{c}{2\pi} + \mu_\alpha \log c + o(\log c), \quad c \to +\infty. \end{aligned}$$

The spectral concentration problem

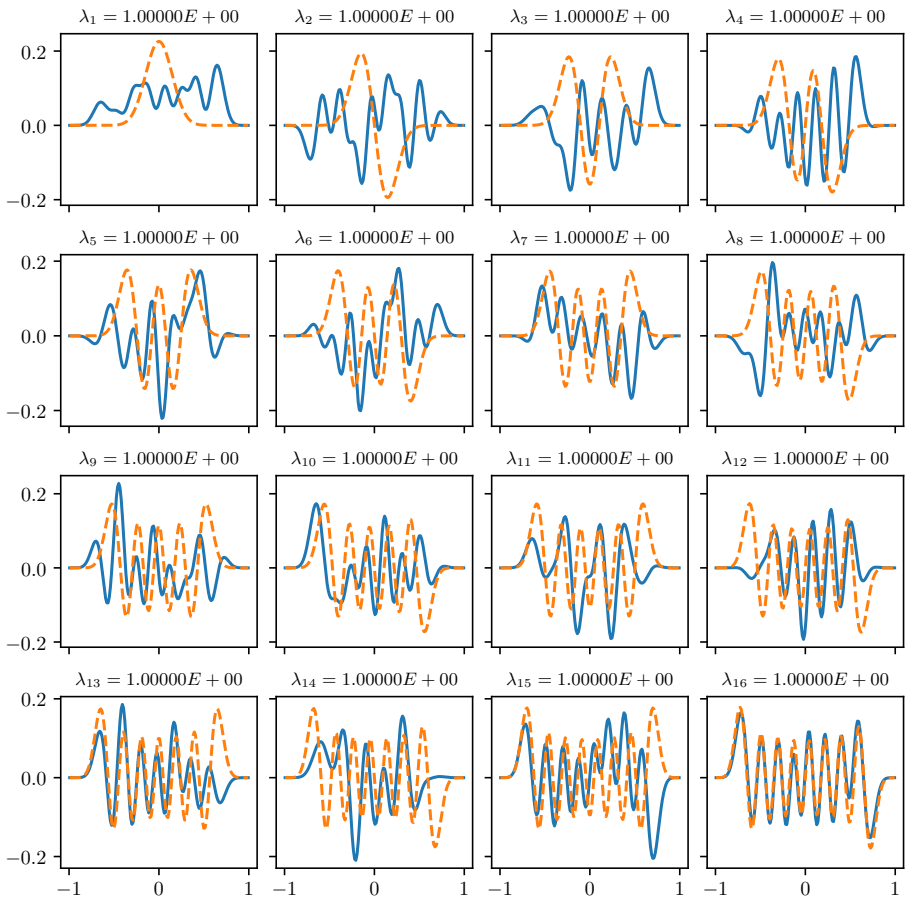
- Obvious extension to rectangles.
- Extension to balls by radial symmetry (Slepian 80s).
- One result with Gaussian filters
  (Brander, De Facio 1986 $\leftarrow$ *9 citations!*)
- Some recent refinements of the Landau-Widom result
  (special functions tools)

That's all folks!

- Constantly used in engineering, optics, geophysics, laser communications, etc...
- The "Slepian basis" seems natural.
  Fast slepian transform, etc...
- Serious numerical problems in computing the basis in higher dimensions.

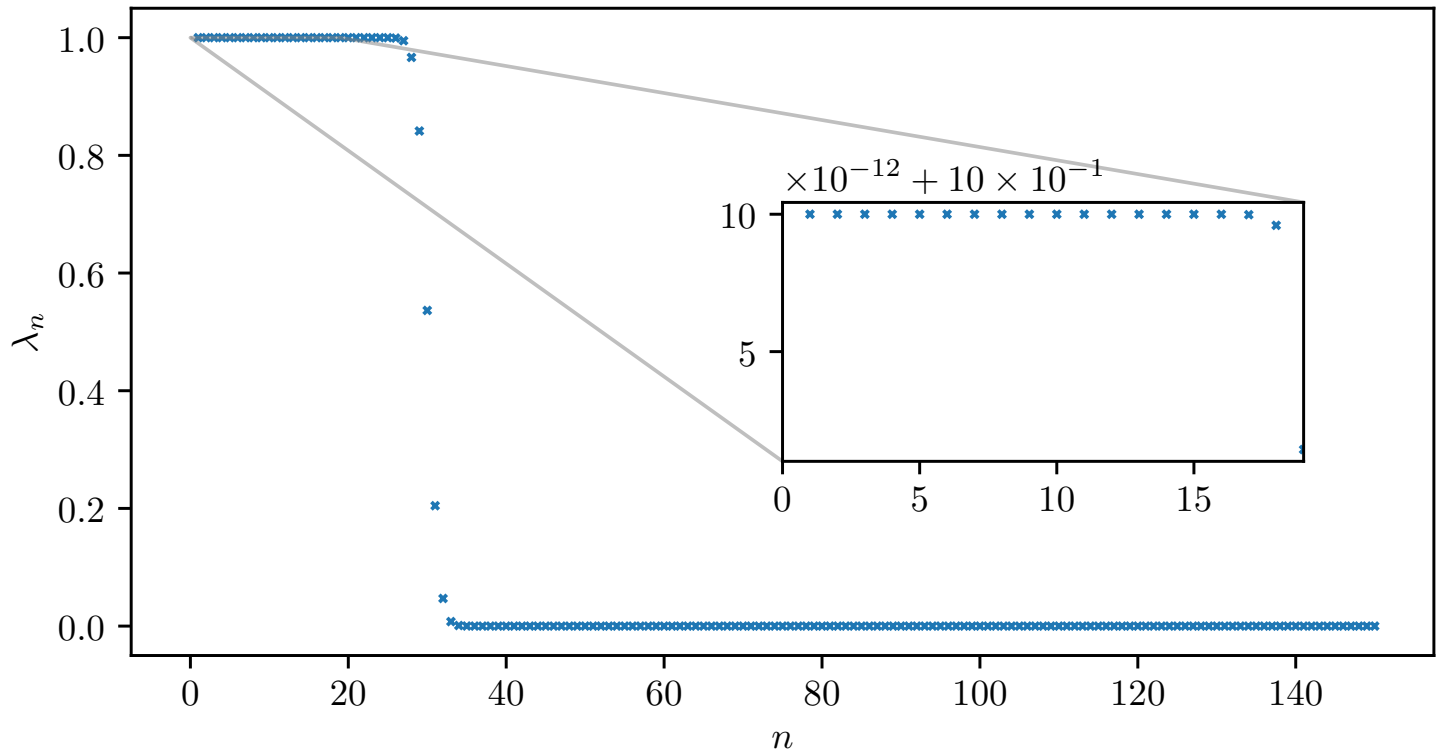
Eigenvectors of the concentration matrix

Eigenvectors of the concentration matrix



Comparison between true eigenvectors (dash orange) and eigenvectors (solid blue) of the discrete concentration matrix ($N = 150$ points). $T = 1$, $W = 0.1 \cdot 2\pi$.

Eigenvalues of the concentration matrix



Eigenvalues of the discrete concentration matrix ($N = 150$ point)
$T = 1$ and $W = 0.1 \cdot 2\pi$.

What can be done apart from the known cases?

Generalized spectral concentration problem

$$\mathcal{F}[f](\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x} dx, \quad \mathcal{F}^{-1}[f](x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\xi) e^{ix\cdot\xi} d\xi.$$

$m_S, \widehat{m_F} \in L^2(\mathbb{R}^d)$ two *masks*, operators on $L^2(\mathbb{R}^d)$:

$$(\mathcal{M}_S f)(x) = m_S(x) f(x), \quad (\mathcal{M}_F f)(x) = \mathcal{F}^{-1}\left[\widehat{m_F} \mathcal{F} f\right](x).$$

Goal: Find maximizers $f \in L^2(\mathbb{R}^d)$ of the *concentration ratio*

$$\nu := \frac{\|\mathcal{M}_F \mathcal{M}_S f\|_2^2}{\|f\|_2^2}.$$

Examples: $m_S = \mathbf{1}_{\Omega_S}$, $\widehat{m_F} = \mathbf{1}_{\Omega_F}$, or m_S and $\widehat{m_F}$ Gaussian functions.

Generalized spectral concentration problem

$$\nu = \frac{\|\mathcal{M}_F \mathcal{M}_S f\|_2^2}{\|f\|_2^2} = \frac{(\mathcal{K}f, f)_{L^2}}{\|f\|_2^2}$$

$$(\mathcal{K}f)(x) = \int_{\mathbb{R}^d} f(y) m_S(y) \overline{m_S(x)} \mathcal{F}^{-1}\left[|\widehat{m_F}|^2\right](x-y)dy, \quad x \in \mathbb{R}^d$$
$$= \overline{m_S(x)} \int_{\mathbb{R}^d} k(x-y) m_S(y) f(y) \mathrm{d}y.$$

- Generalized Spectral concentration problem: Find eigenpairs of $\mathcal{K}$

$$\mathcal{K}\psi = \lambda\psi.$$

- Case of domains:

$$(\mathcal{K}f)(x) = \int_{\Omega_S} k(x-y) f(y) \mathrm{d}y, \quad x \in \Omega_S$$
$$k(z) = \int_{\Omega_F} e^{iz\xi} \mathrm{d}\xi.$$

Properties:

$$\|\mathcal{K}\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \leq \|m_S\|_2^2 \|\widehat{m_F}\|_2^2.$$

1. The concentration operator $\mathcal{K}$ is a *Hilbert-Schmidt* operator, self-adjoint, compact, and positive semi-definite.
2. The countable family $\{\psi_i\}_{i=1}^{\infty}$ of eigenfunctions of $\mathcal{K}$ is orthonormal and complete in $L^2(\mathbb{R}^d)$.
   Associated eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ real, nonnegative, $\lambda_i \geq \lambda_{i+1} \geq 0$.
3. min-max theorems.
4. $\lambda_n = o(n^{-1/2})$, $n \to +\infty$.

Symmetries:

$\mathbf{S} \in \mathbb{R}^{d\times d}$ orthogonal matrix. If

$$m_S \circ \mathbf{S} = \alpha m_S, \quad |\alpha| = 1$$
$$|\widehat{m_F} \circ \mathbf{S}| = |\widehat{m_F}|\,.$$

Then

$$\left| \begin{array}{l} \mathcal{K}\psi = \lambda\psi \\ \lambda \quad \text{multiplicity} \quad 1 \end{array} \right. \qquad \Longrightarrow \quad \psi \circ \mathbf{S} = \beta\psi, \quad |\beta| = 1.$$

Translations with binary masks

Lemma

Let $\Omega_S, \Omega_F \subset \mathbb{R}^d$, $p \in \mathbb{R}^d$. The following equivalences hold:

- *(λ, ψ) is an eigenpair of the concentration operator associated to masks $m_S = \mathbf{1}_{\Omega_S + p}$ and $\widehat{m_F} = \mathbf{1}_{\Omega_F}$ iff $(\lambda, \psi \circ \tau_{-p})$ is an eigenpair of the concentration operator associated to masks $m_S = \mathbf{1}_{\Omega_S}$ and $\widehat{m_F} = \mathbf{1}_{\Omega_F}$;*
- *(λ, ψ) is an eigenpair of the concentration operator associated to masks $m_S = \mathbf{1}_{\Omega_S}$ and $\widehat{m_F} = \mathbf{1}_{\Omega_F + p}$ iff $(\lambda, x \mapsto \psi(x) e^{-ip \cdot x})$ is an eigenpair of the concentration operator associated to masks $m_S = \mathbf{1}_{\Omega_S}$ and $\widehat{m_F} = \mathbf{1}_{\Omega_F}$.*

Affine transformations with binary masks

Lemma

Let $\mathbf{A} \in \mathbb{R}^{d \times d}$ *an invertible matrix,* $\Omega_S, \Omega_F \subset \mathbb{R}^d$*, and write*

$$\mathbf{A}\Omega_F := \{\mathbf{A}z : z \in \Omega_F\}.$$

Let (λ, ψ) *an eigenpair of the concentration operator associated to binary masks* $m_S = \mathbf{1}_{\Omega_S}$ *and* $\widehat{m_F} = \mathbf{1}_{\mathbf{A}\Omega_F}$*. Then,* $(\lambda, \psi \circ \mathbf{A}^{-T})$ *is an eigenpair of the concentration operator associated to binary masks* $m_S = \mathbf{1}_{\mathbf{A}^T\Omega_S}$ *and* $\widehat{m_F} = \mathbf{1}_{\Omega_F}$*. The converse is also true.*

Exact eigenvectors for quadric binary masks

Let

$$D(\mathbf{M}, v, c) := \left\{ x \in \mathbb{R}^d : x^T \mathbf{M} x + v^T x + c \leq 0 \right\}.$$

Theorem

Let $\Omega_S = D(\mathbf{M}_S, v_s, c_S)$ and $\Omega_F = D(\mathbf{M}_F, v_F, c_F)$, for some symmetric, diagonalizable and invertibles matrices $\mathbf{M}_S, \mathbf{M}_F \in \mathbb{R}^{d \times d}$, vectors $v_S, v_F \in \mathbb{R}^d$ and scalars $c_S, c_F \in \mathbb{R}$. Let $\mathcal{K}$ the concentration operator associated to masks $m_S = \mathbf{1}_{\Omega_S}$ and $\widehat{m_F} = \mathbf{1}_{\Omega_F}$. Then, there exists a second-order differential operator $\mathcal{P}$ commuting with $\mathcal{K}$.

Exact eigenvectors for quadric binary masks

$\mathbf{M}_S = U_S \Lambda_S U_S^T$, $\mathbf{M}_F = U_F \Lambda_F U_F^T$, where $U_S, U_F \in \mathbb{R}^d$ are orthogonal matrices and Λ_S, Λ_F are diagonal.
Let $a_n = (\Lambda_S)_{n,n}$, $\alpha_n = (\Lambda_F)_{n,n}$. Let $w_S = -\frac{1}{2}\Lambda_S^{-1} U_S^T v_S$, $w_F = -\frac{1}{2}\Lambda_F^{-1} U_F^T v_F$, $b = w_S^T \Lambda_S w_S - c_S$, $\beta = w_F^T \Lambda_F w_F - c_F$.
Then,

$$\mathcal{P}(x, \nabla_x) = \nabla_x^T U^T \mathbf{A}\left(U_S^T x + \frac{1}{2}\Lambda_S^{-1} U_S^T v_s\right) U^T \nabla_x + C\left(U_S^T x + \frac{1}{2}\Lambda_S^{-1} U_S^T v_s\right)$$

where

$$\mathbf{A}(y) := \operatorname{diag}\left\{\alpha_m \left(\sum_{n=1}^{d} a_n y_n^2 - b\right)\right\}_{m=1}^{d} \quad \text{and} \quad C(y) := \beta y \cdot \operatorname{diag}\left\{a_n\right\}_{n=1}^{d} y.$$

A splitting approach

$V, H \geq 0$ two functions.

$$m_S(x) = e^{-V(x)} \quad \text{and} \quad \widehat{m_F}(\xi) = e^{-H(\xi)}$$

Then

$$\mathcal{K} = e^{-V(x)} e^{-H(-i\partial_x)} e^{-V(x)}$$

Formally: Baker-Campbell-Hausdorff formula

$$e^{-V(x)} e^{-H(-i\partial_x)} e^{-V(x)} = e^{-Z(x,\partial_x)}$$

$$Z = H + 2V - \frac{1}{12}[H,[H,V]] + \frac{1}{3}[V,[V,H]] + \cdots$$

In general no convergence, possible sign issues...

A splitting approach

In case of quadratic H and V it is known.

- Formula (P. Alphonse, J. Bernier 2022)

$$e^{-\frac{1}{2}\tanh(z)x^2}\, e^{\frac{1}{2}\sinh(2z)\Delta}\, e^{-\frac{1}{2}\tanh(z)x^2} = e^{-z(x^2-\Delta)}$$

- Implies by scaling

$$e^{-\frac{1}{2}\sinh(z)x^2}\, e^{\sinh(z)\Delta}\, e^{-\frac{1}{2}\sinh(z)x^2} = e^{-z(\cosh(z)x^2-\frac{1}{\cosh(z)}\Delta)}$$

$$e^{-\frac{1}{2}cx^2}\, e^{c\Delta}\, e^{-\frac{1}{2}cx^2} = e^{-\mathrm{argsh}(c)[\sqrt{1+c^2}x^2-\frac{1}{\sqrt{1+c^2}}\Delta)}$$

- Explicit spectrum

$$\mathrm{spec}\{-\frac{1}{\mu^2}\Delta+\mu^2x^2\} = \{2n+1, \varphi_n(\mu x)\}_{n\in\mathbb{N}},$$

φ_n Hermite functions.

A splitting approach

Spectrum of $e^{-\frac{1}{2}cx^2} e^{c\Delta} e^{-\frac{1}{2}cx^2} \psi_n = \lambda_n \psi_n$

$$\psi_n = (1+c^2)^{\frac{1}{8}} \varphi_n\big((1+c^2)^{\frac{1}{4}} x\big), \quad (\varphi_n \text{ Hermite functions})$$
$$\lambda_n = e^{-\operatorname{argsh}(c)(2n+1)}$$

Theorem

If $m_S = e^{-\frac{\alpha}{2}x^2}$, $\widehat{m_F} = e^{-\frac{\beta}{2}\xi^2}$ *then* $\mathcal{K}\psi_n = \lambda_n \psi_n$ *with*

$$\psi_n = \frac{\beta^{\frac{1}{4}}(1+\alpha\beta)^{\frac{1}{8}}}{\alpha^{\frac{1}{4}}} \varphi_n\Big(\sqrt{\frac{\beta}{\alpha}}(1+\alpha\beta)^{\frac{1}{4}} x\Big)$$
$$\lambda_n = e^{-\operatorname{argsh}(\sqrt{\alpha\beta})(2n+1)}$$

and φ_n *Hermite functions.*

Easy generalization to higher dimension, more general quadratic operators.

Numerical simulations

Discretization of the concentration operator $\mathcal{K}$ using N uniform points along each dimension $\rightarrow$ concentration matrix $\mathbf{K} \in \mathbb{C}^{N^d \times N^d}$.
Eigenvectors of $\mathbf{K}$.

1. Hermitian character: $\mathbf{K}^* = \mathbf{K}$.
2. Structure: $\mathbf{K} = D^* B D$, with B a block matrix where each block is Toeplitz, and D is a diagonal matrix.
3. Its eigenvalues are real.
4. Its eigenvectors form an unitary basis of $\mathbb{C}^{N^d}$.

Numerical remark

In 2d, we have

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}^{(0,0)} & \mathbf{K}^{(0,1)} & \dots & \mathbf{K}^{(0,N-2)} & \mathbf{K}^{(0,N-1)} \\ \mathbf{K}^{(1,0)} & \mathbf{K}^{(1,1)} & \dots & \mathbf{K}^{(1,N-2)} & \mathbf{K}^{(1,N-1)} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \mathbf{K}^{(N-2,0)} & \mathbf{K}^{(N-2,1)} & \dots & \mathbf{K}^{(N-2,N-2)} & \mathbf{K}^{(N-2,N-1)} \\ \mathbf{K}^{(N-1,0)} & \mathbf{K}^{(N-1,1)} & \dots & \mathbf{K}^{(N-1,N-2)} & \mathbf{K}^{(N-1,N-1)} \end{pmatrix},$$

where each $\mathbf{K}^{(m,n)}$ is a Toeplitz matrix of size $N \times N$.

For $j, k \in [\![0, N-1]\!]^d$,

$$\mathbf{K}_{j,k} = \frac{m_S(x^{(k)})\overline{m_S(x^{(j)})}}{(2N-1)^d} \sum_{l \in [\![0,2N-2]\!]^d} \left|\widehat{m_F}(\xi^{(l)})\right|^2 e^{i\frac{2\pi}{2N-1} l\cdot(j-k)}.$$

New numerical algorithm

Using the a priori bounds: As $c \to 0$, $\lambda_j \to 0$.

For $\omega > 0$, consider the masks

$$m_S(x) = \mathbf{1}_{[-1,1]}(x) \quad \text{and} \quad \widehat{m_F}(x) = \mathbf{1}_{[-\omega,\omega]},$$

and their parameterized versions

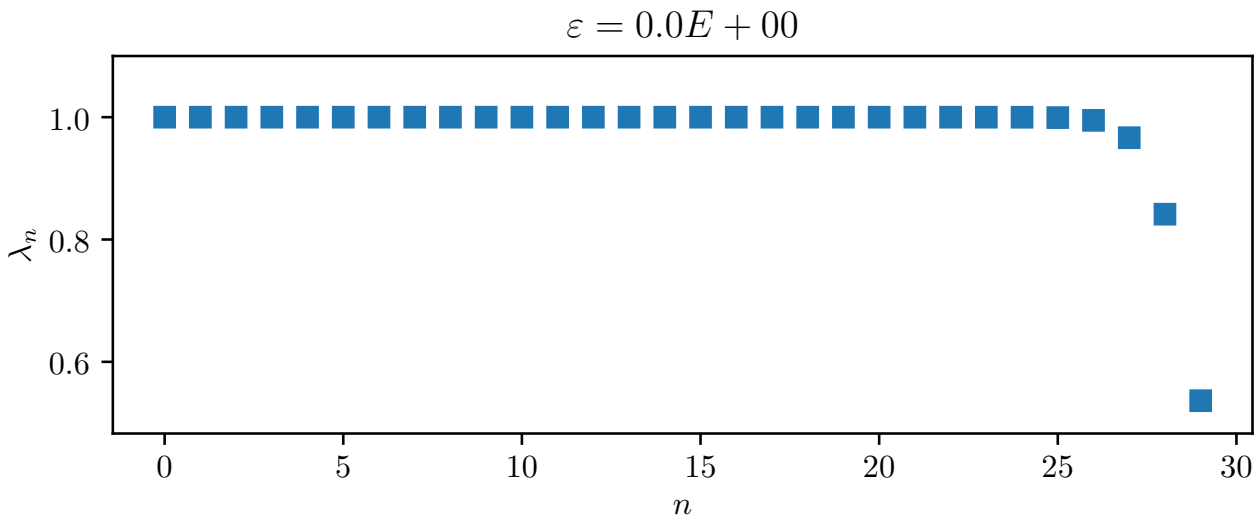
$$m_S(\varepsilon, x) = \mathbf{1}_{[-\mu(\varepsilon),\mu(\varepsilon)]}(x) \quad \text{and} \quad \widehat{m_F}(\varepsilon, x) = \mathbf{1}_{[-\mu(\varepsilon)\omega,\mu(\varepsilon)\omega]},$$

where we impose $\mu : \mathbb{R}_+ \to [0, 1]$ nonincreasing with

$$\mu(\varepsilon) \to \begin{cases} 1 & \text{as } \varepsilon \to 0 \\ 0 & \text{as } \varepsilon \to \infty. \end{cases}$$

It gives new operators $\mathcal{K}(\varepsilon)$ and new matrices $\mathbf{K}(\varepsilon)$.

Eigenvalues of $\mathbf{K}(\varepsilon)$ as $\varepsilon \to +\infty$



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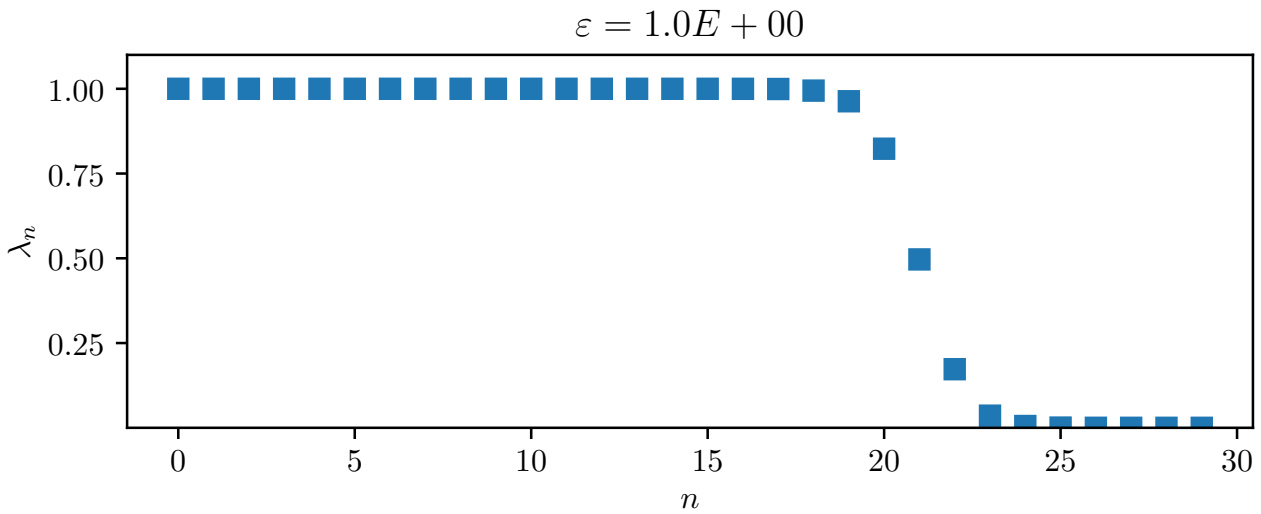

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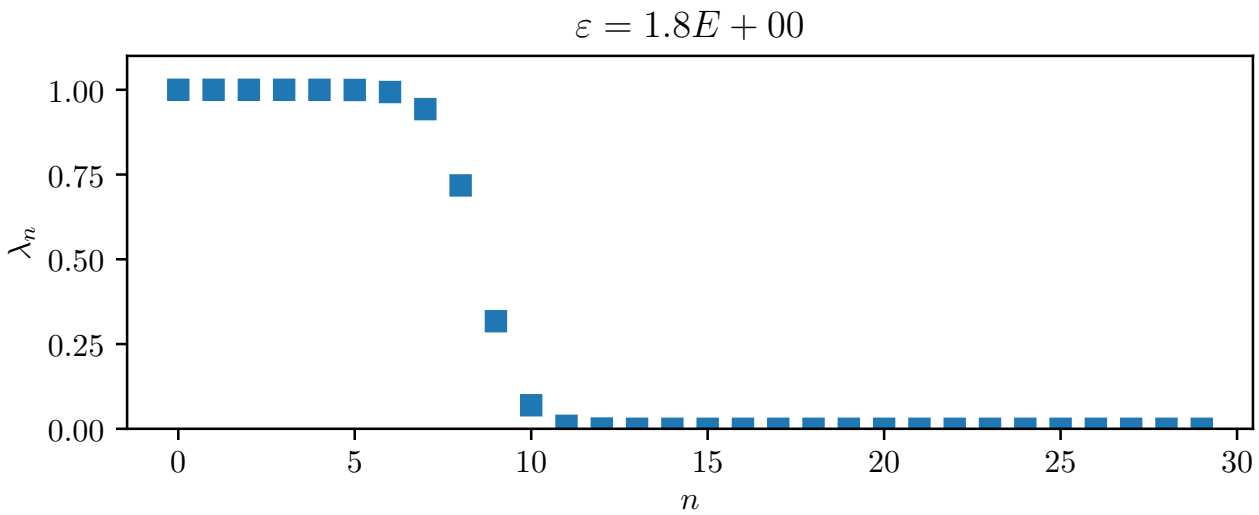

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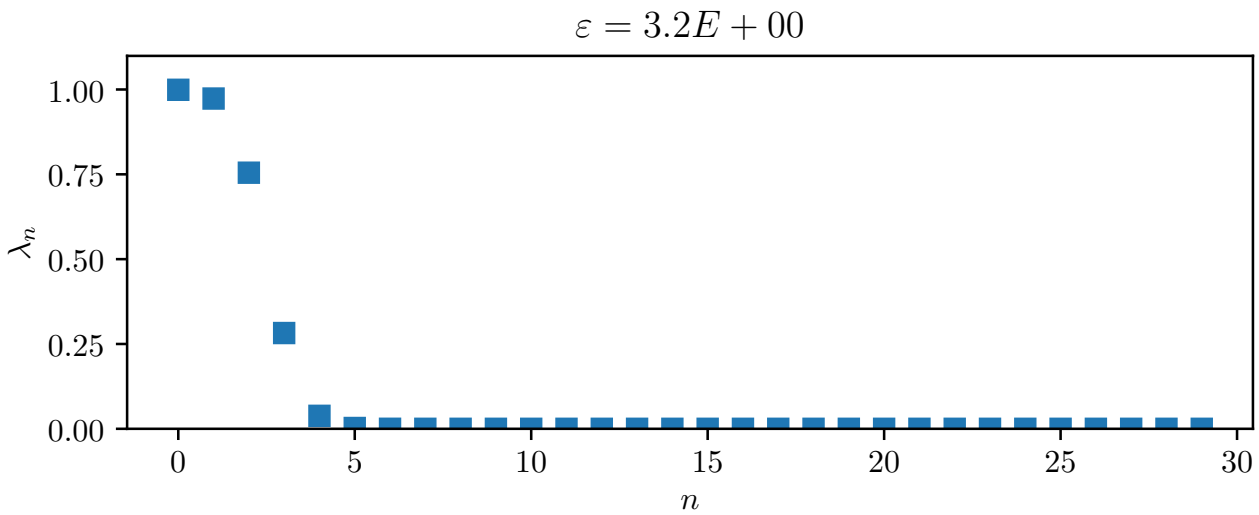
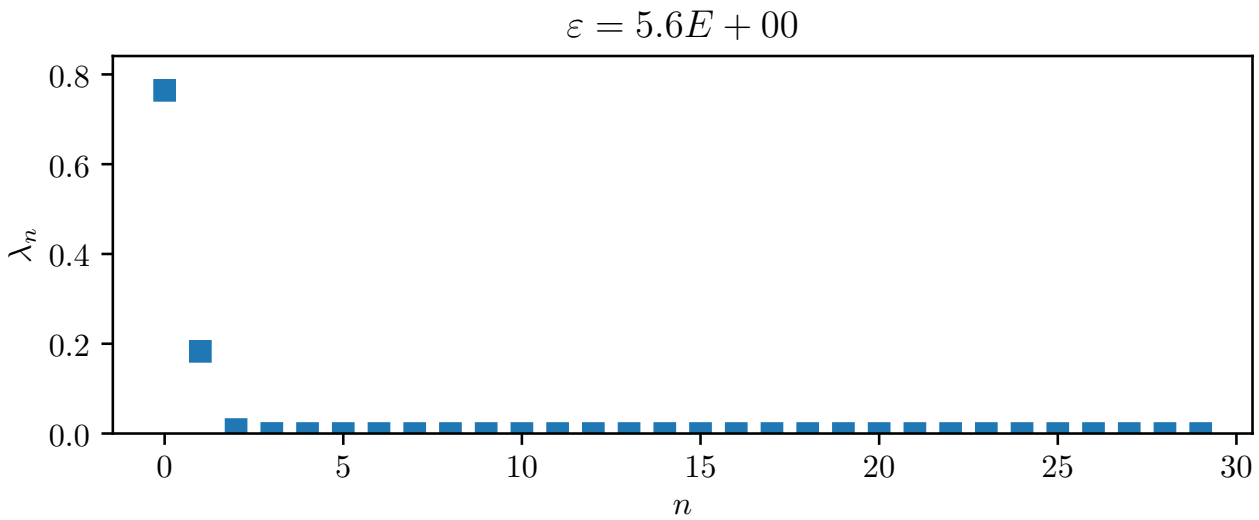


Eigenvalues of $\mathbf{K}(\varepsilon)$ as $\varepsilon \to +\infty$



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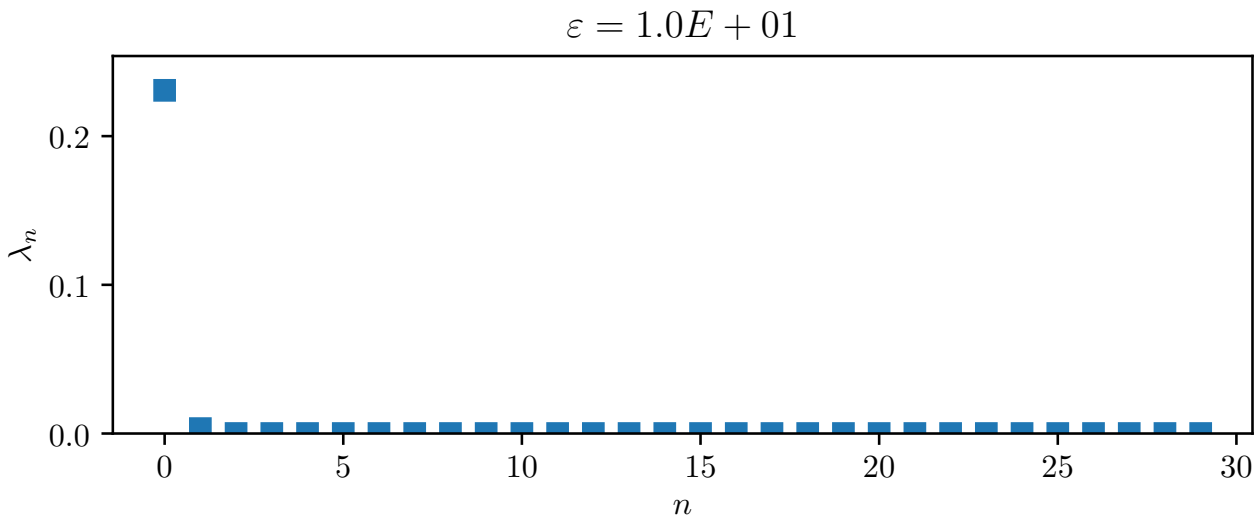
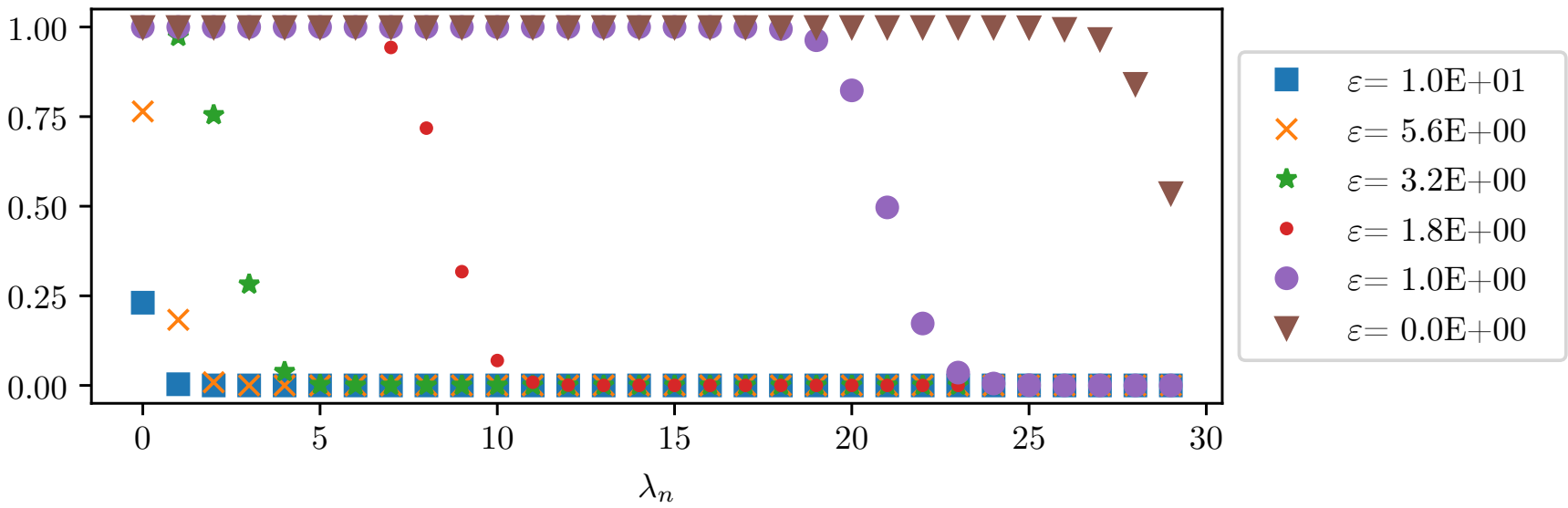


Eigenvalues of $\mathbf{K}(\varepsilon)$ as $\varepsilon \to +\infty$

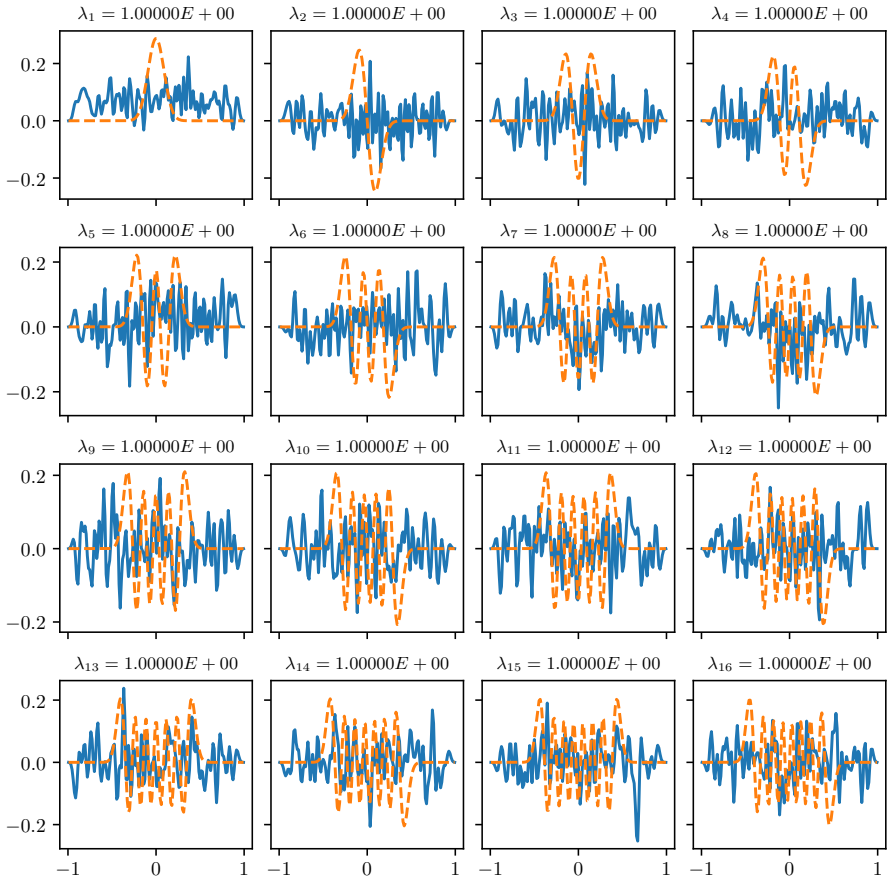


Summary: by varying ε, we are able to "separate" the eigenvalues.

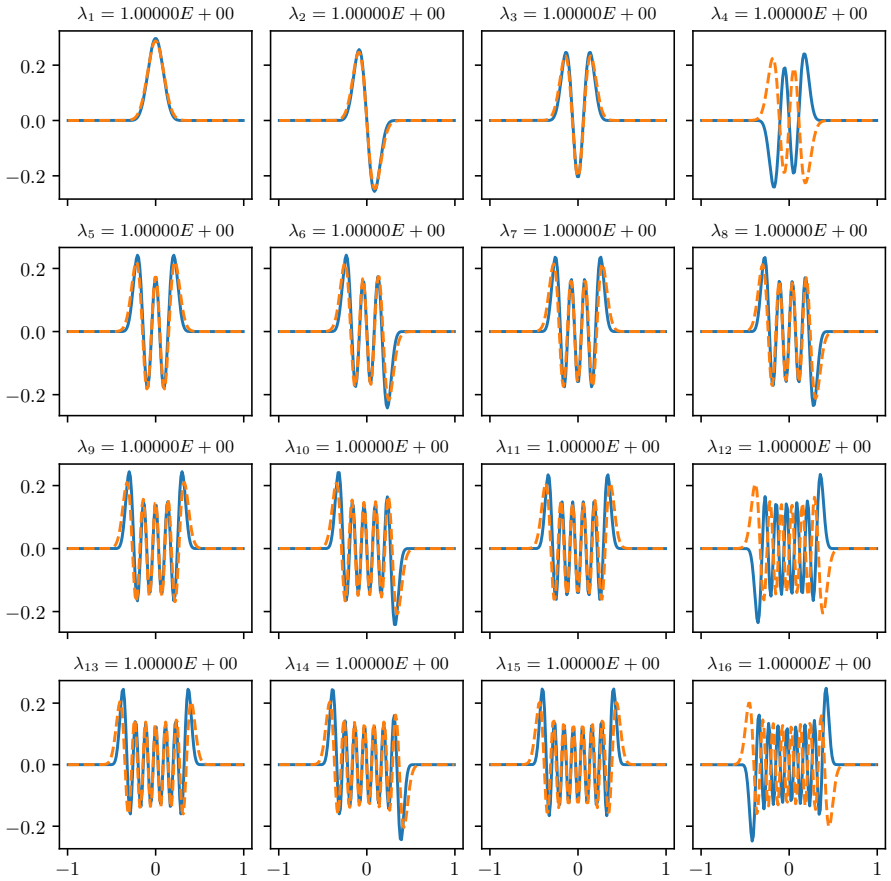
Outline of the algorithm:

- take ε large enough so that the first eigenvalue is distinct from the others, and obtain its eigenvector. Check if it is a good approximation of the true eigenvector (for the nonperturbed problem, based on some numerical tolerance η).
- If so, do the same process on the next eigenpair. If not, take ε a little smaller and repeat the process.
- In practice, one of the best scaling is

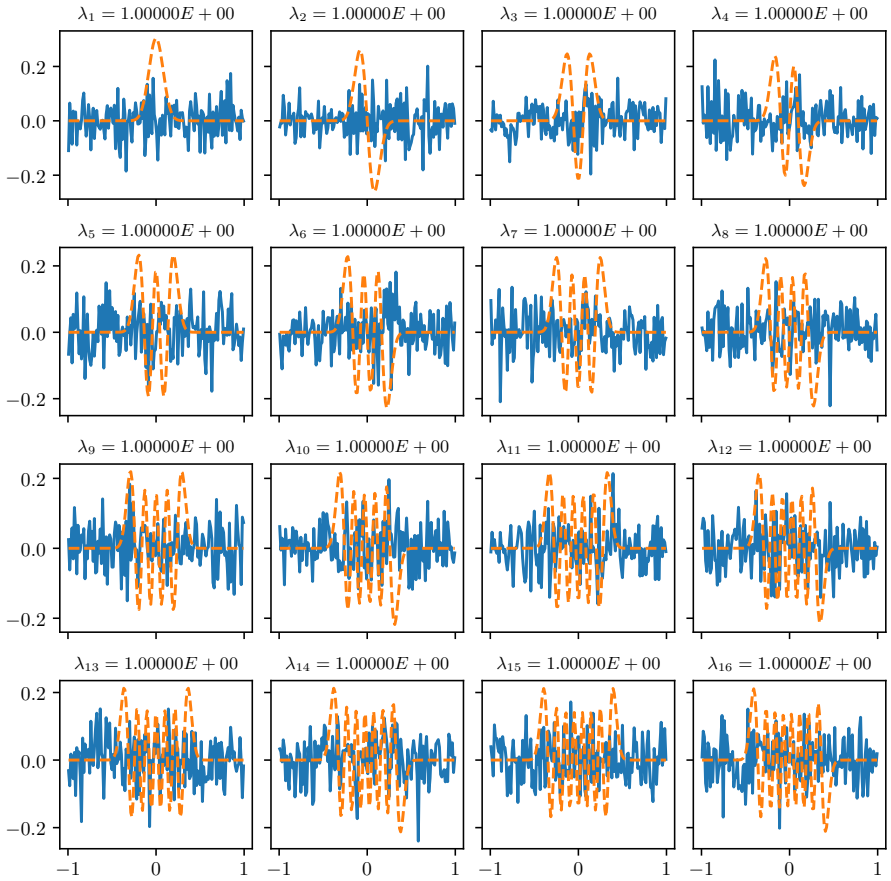
$$\mu(\varepsilon) := \frac{1}{(1+\varepsilon^4)^{1/4}}.$$



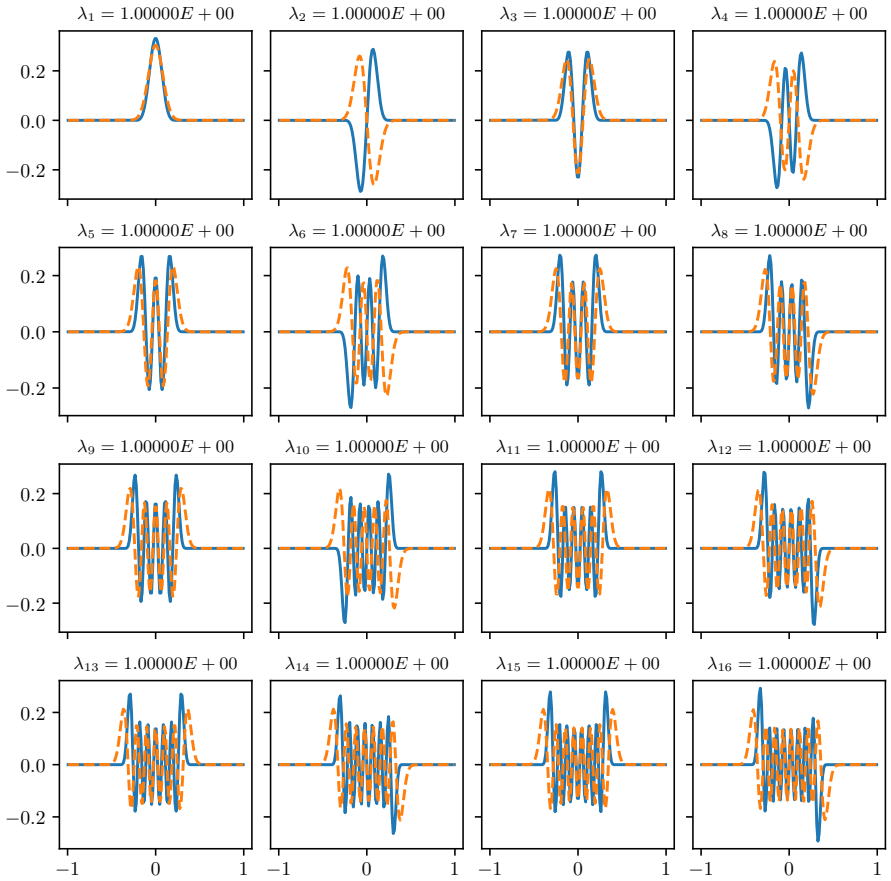
(a) 16 first eigenvectors obtained with an eigendecomposition (solid blue curve) of $\mathbf{K}(0)$, compared to the exact eigenvectors (dashed orange curve). Here, $\widehat{m_F} = \mathbf{1}_{[-0.3\cdot 2\pi\mu(\varepsilon), 0.3\cdot 2\pi\mu(\varepsilon)]}$. $N_1 = 150, \eta = 10^{-10}$.



(b) 16 first eigenvectors obtained with the varying masks procedure (solid blue curve) of $\mathbf{K}(0)$, compared to the exact eigenvectors (dashed orange curve). Here, $\widehat{m_F} = \mathbf{1}_{[-0.3\cdot 2\pi\mu(\varepsilon), 0.3\cdot 2\pi\mu(\varepsilon)]}$. $N_1 = 150, \eta = 10^{-10}$.

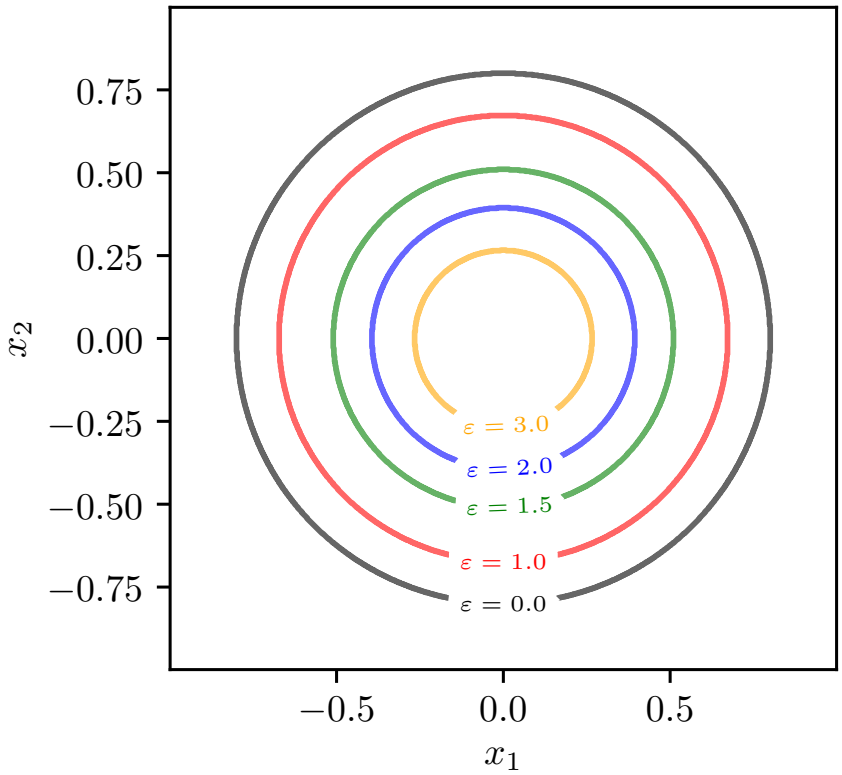


(a) 16 first eigenvectors obtained with an eigendecomposition (solid blue curve) of $\mathbf{K}(0)$, compared to the exact eigenvectors (dashed orange curve). Here, $\widehat{m_F} = \mathbf{1}_{[-0.49 \cdot 2\pi\mu(\varepsilon), 0.49 \cdot 2\pi\mu(\varepsilon)]}$. $N_1 = 150, \eta = 10^{-10}$.

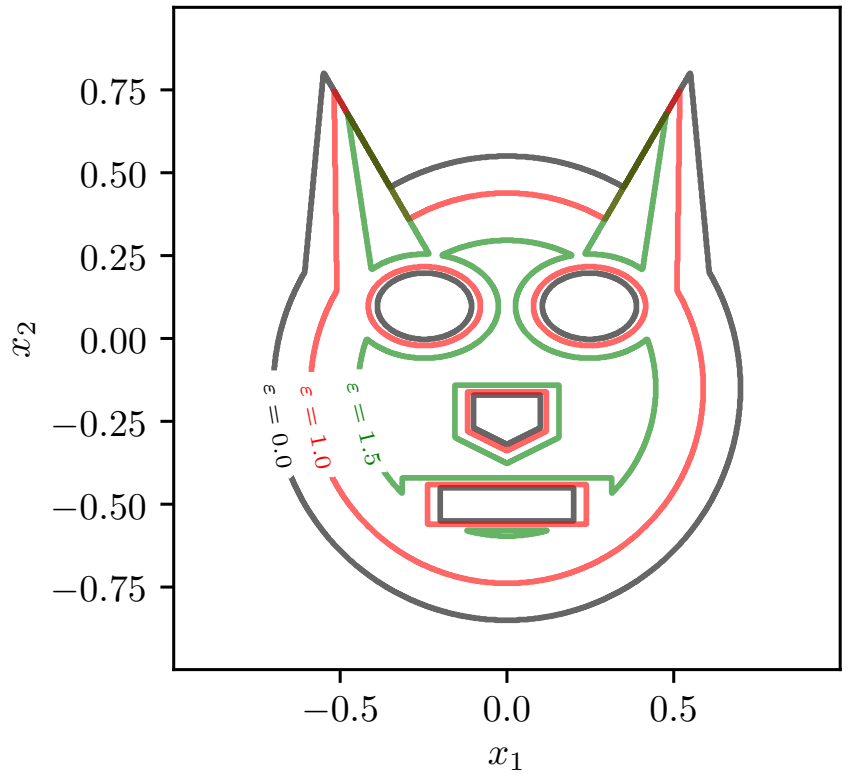


(b) 16 first eigenvectors obtained with the varying masks procedure (solid blue curve) of $\mathbf{K}(0)$, compared to the exact eigenvectors (dashed orange curve). Here, $\widehat{m_F} = \mathbf{1}_{[-0.49 \cdot 2\pi\mu(\varepsilon), 0.49 \cdot 2\pi\mu(\varepsilon)]}$. $N_1 = 150, \eta = 10^{-10}$.

We wish to apply this varying mask idea in 2d

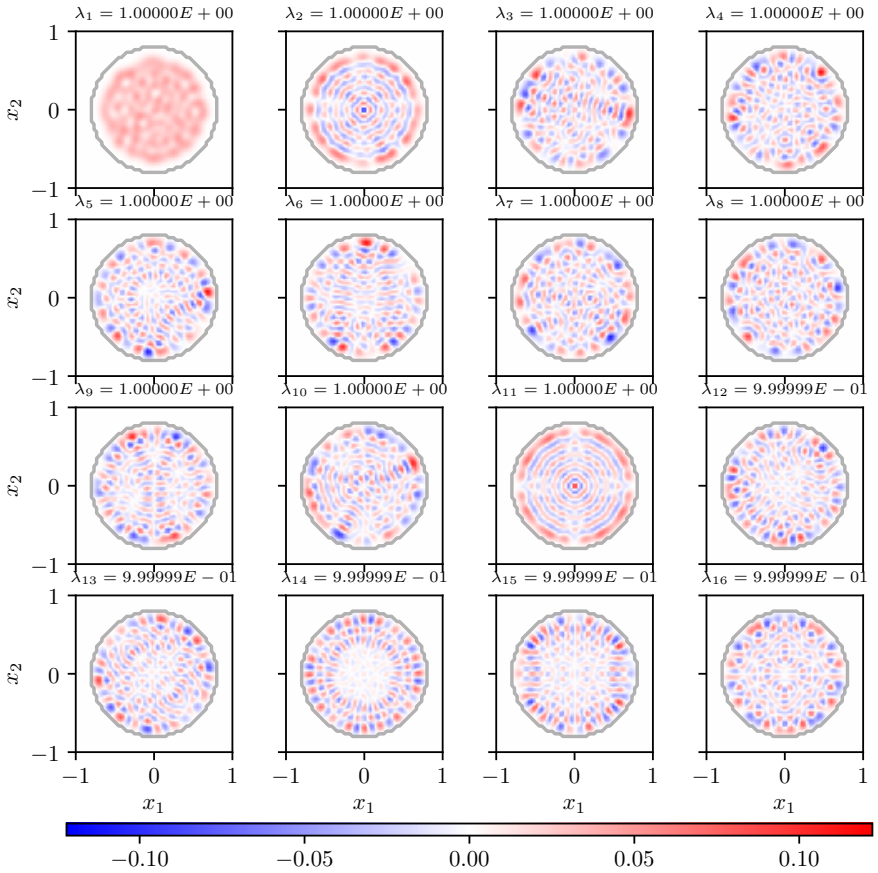


(a) Scaling of a disc as ε varies.

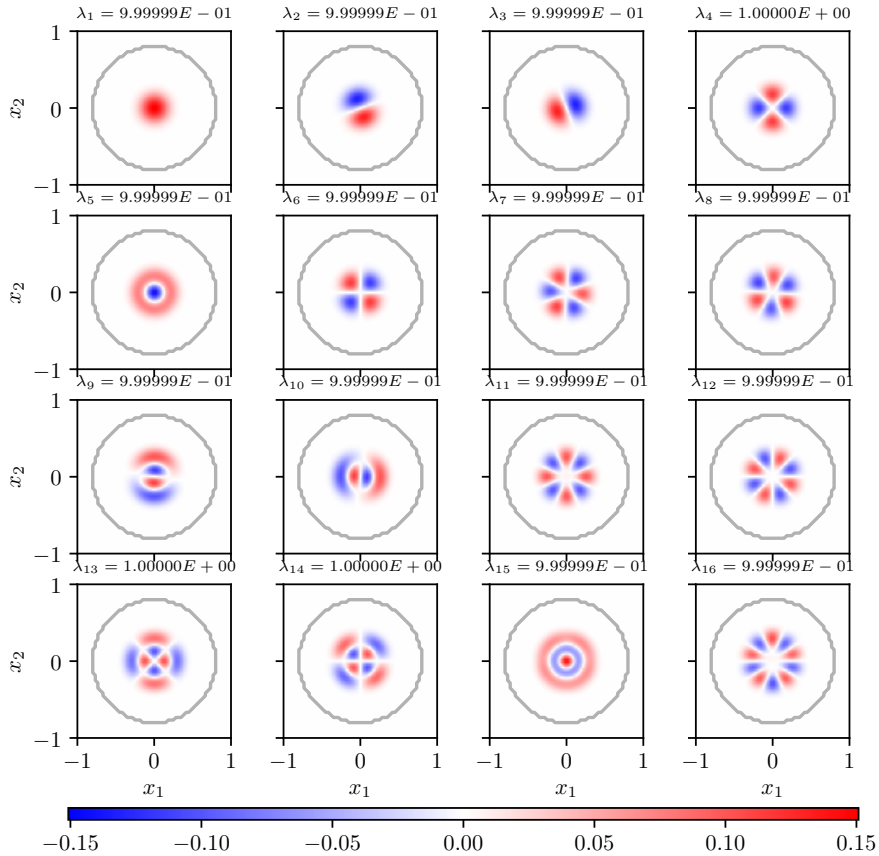


(b) Scaling of cat-head domain as ε varies. Attention is paid to having the holes (eyes, nose, mouth) only be scaled and to not move.

$m_S = \mathbf{1}_{\mathrm{Disc}(0,0.8)}$ $\widehat{m_F} = \mathbf{1}_{\mathrm{Disc}(0,0.3\cdot 2\pi)}$

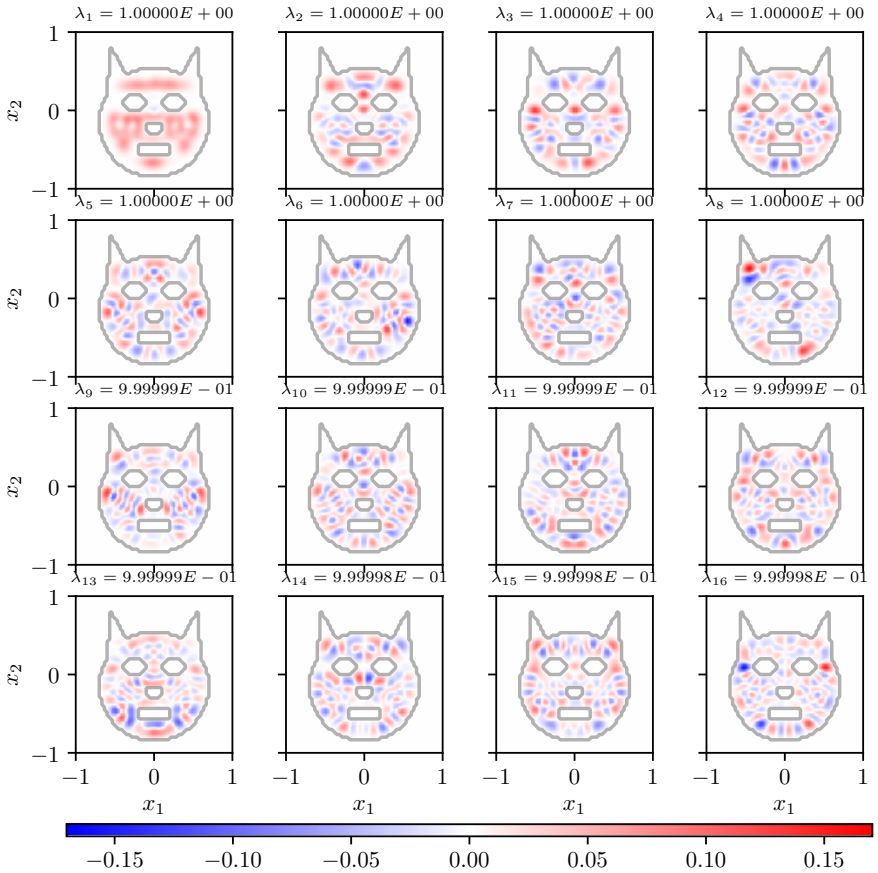


(a) Eigenvectors obtained with a standard eigendecomposition of the initial concentration matrix $\mathbf{K}(0)$

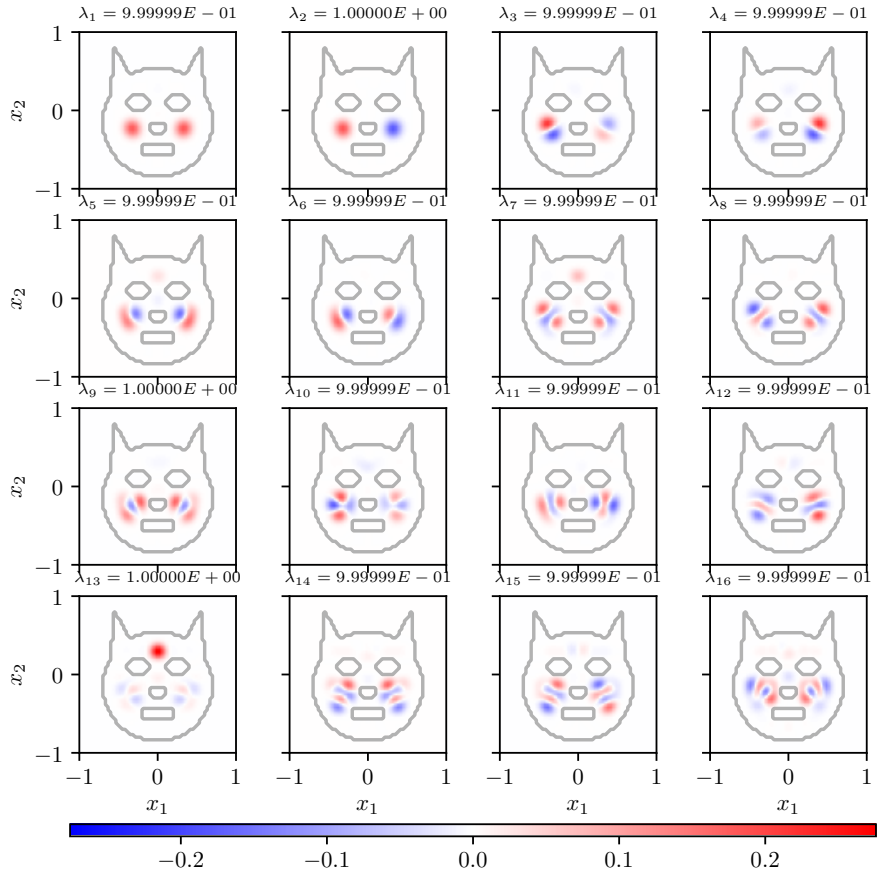


(b) Eigenvectors obtained with an approximate eigendecomposition obtained with the varying mask algorithm.

$$m_S = \mathbf{1}_{\text{cat}-\text{head}} \qquad \widehat{m_F} = \mathbf{1}_{\text{Disc}(0,0.3\cdot 2\pi)}$$

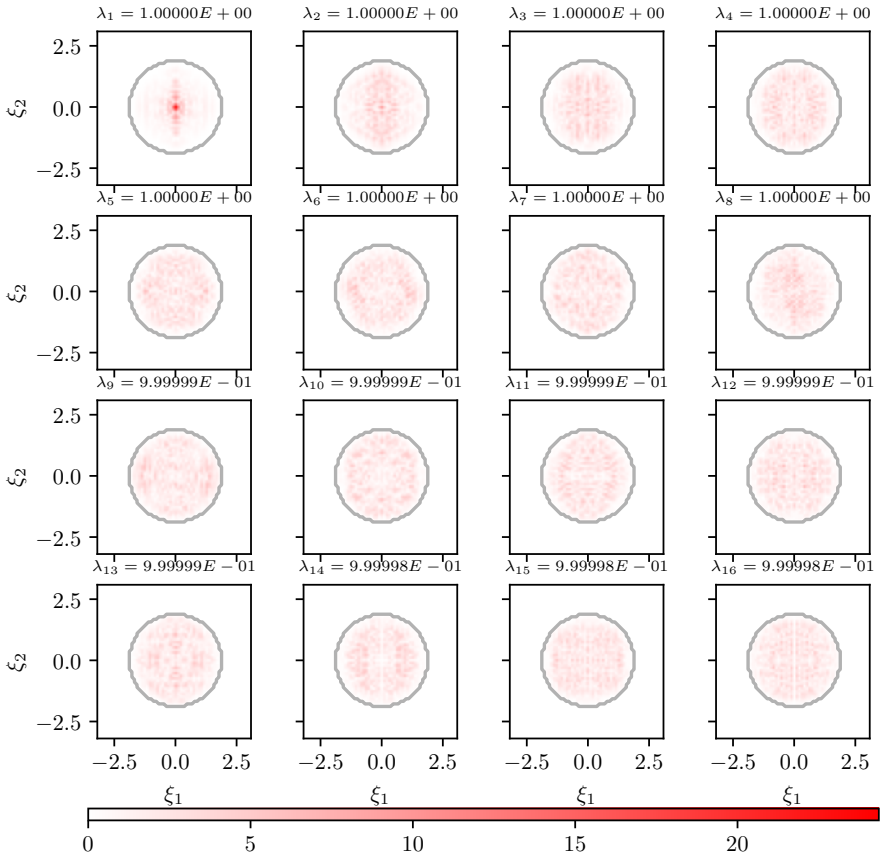


(a) 16 first eigenvectors obtained with a standard eigendecomposition of the initial concentration matrix $\mathbf{K}(0)$

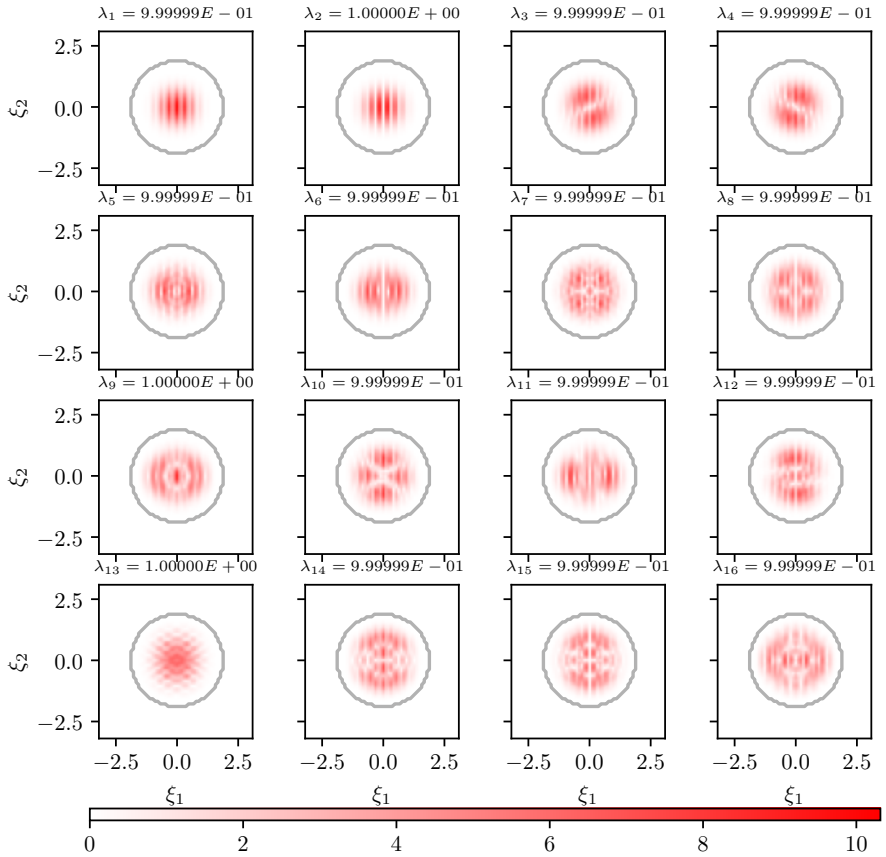


(b) 16 first eigenvectors obtained with an approximate eigendecomposition obtained with the varying mask algorithm.

$m_S = \mathbf{1}_{\text{cat}-\text{head}}$ $\widehat{m_F} = \mathbf{1}_{\text{Disc}(0,0.3\cdot 2\pi)}$



(a) Fourier transform of the 16 first eigenvectors obtained with a standard eigendecomposition of the initial concentration matrix $\mathbf{K}(0)$



(b) Fourier transform of the 16 first eigenvectors obtained with an approximate eigendecomposition obtained with the varying mask algorithm.

Summary of the algorithm

Advantages:

- the approximate eigenvectors obtained are a unitary basis
- the size of the approximation is controlled using η
- it relies on standard eigendecomposition applied to modified matrices, so the block-Toeplitz nature of $\mathbf{K}$ can be used
- the scaling idea can be adapted to multidimensional situations, and it also yielded good results in tested situations
- the expected features are recovered (symmetry and localization)

Disadvantages:

- this "shrinking and expanding" idea has worked for all our numerical experiments, but there is NO guarantee that we can always separate the eigenvalues enough as $\varepsilon \to +\infty$ (even in 1d, it is not guaranteed!)
- the choice of μ is entirely arbitrary, with no idea what would be better
- there is no proof, nor intuition, that having the same scaling $\mu(\varepsilon)$ in space and Fourier is the best choice

Conclusion

We propose an algorithm to approximate the eigenvectors of the discretized spectral concentration operator $\mathcal{K}$, and it yields satisfying results with expected features.
However, there is no rigorous proof that it should work, nor that the assumptions used always hold.

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Thank you!