Modulation algorithm for the Schrödinger equation

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Joint work with E. Faou and P. Raphael

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Schrödinger equation

Cubic NLS:

$$\begin{cases} i\partial_t \psi + \Delta_x \psi - |x|^2 \psi = |\psi|^2 \psi, \\ \psi(t=0) = \psi_0, \end{cases} \tag{cNLS}$$

$\psi \equiv \psi(t,x)$, $t \in \mathbb{R}^*$, $x \in \mathbb{R}^d$.

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- Harmonic oscillator:

$$i\partial_t \psi + \Delta_x \psi - |x|^2 \psi = 0 \tag{HO}$$

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Made of two parts

- Harmonic oscillator:

$$i\partial_t\psi + \Delta_x\psi - |x|^2\psi = 0 \tag{HO}$$

- Cubic nonlinearity:

$$i\partial_t\psi = |\psi|^2\psi \tag{1.1}$$

Time-splitting

$$\psi_0 \rightarrow [(\text{cNLS}), \Delta t] \rightarrow \psi(t = \Delta t),$$

Time-splitting

$$\psi_0 \to [(\text{cNLS}), \Delta t] \to \psi(t = \Delta t),$$

$$\psi_0 \to [(\text{HO}), \Delta t] \to [(1.1), \Delta t] \to \phi(t = \Delta t),$$

Then

$$\phi(\Delta t) = \psi(\Delta t) + \mathcal{O}(\Delta t)$$

Time-splitting

$$\psi_0 \to [(\text{cNLS}), \Delta t] \to \psi(t = \Delta t),$$

$$\psi_0 \to [(\text{HO}), \Delta t] \to [(1.1), \Delta t] \to \phi(t = \Delta t),$$

$$\psi_0 \to [(\text{HO}), \frac{1}{2}\Delta t] \to [(1.1), \Delta t] \to [(\text{HO}), \frac{1}{2}\Delta t] \to \varphi(t = \Delta t),$$

Then

$$\phi(\Delta t) = \psi(\Delta t) + \mathcal{O}(\Delta t)$$
$$\varphi(\Delta t) = \psi(\Delta t) + \mathcal{O}(\Delta t^2)$$

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Modulation

"Bubbles" introduced in[1,2], to study blow-up of solutions.
Let

$$\psi(t,x) \approx u(t,x) := \sum_{j=1}^{N} u_j(t,x),$$

with

$$u_j(t,x) := \frac{A_j}{L_j} e^{i\gamma_j + iL_j\beta_j\cdot y_j - i\frac{B_j}{4}|y_j|^2} v_j(s_j, y_j), \quad \text{with} \quad \left|\begin{aligned} \frac{\mathrm{d}s_j}{\mathrm{d}t} &:= \frac{1}{L_j^2}, \\ y_j &:= \frac{x - X_j}{L_j}, \end{aligned}\right.$$

[1] Yvan Martel and Pierre Raphaël. "Strongly Interacting Blow up Bubbles for the Mass Critical Nonlinear Schrödinger Equation". In: *Annales scientifiques de l'École normale supérieure* 51.3 (2018), pp. 701–737. ISSN: 0012-9593, 1873-2151. DOI: 10.24033/asens.2364.

[2] Erwan Faou and Pierre Raphael. "On Weakly Turbulent Solutions to the Perturbed Linear Harmonic Oscillator". In: *arXiv:2006.08206 [math]* (June 2020). arXiv: 2006.08206 [math]. URL: http://arxiv.org/abs/2006.08206.

Plug previous ansatz in the Harmonic Oscillator (omit j subscript):

$$\begin{aligned}
&i\partial_t u + \Delta_x u - |x|^2 u \\
&\quad = \frac{A}{L^3} e^{i\gamma + iL\beta\cdot y - i\frac{B}{4}|y|^2} \Big\{ i\partial_s v + \left(-\gamma_s + \beta\cdot X_s - L^2\left(|\beta|^2 + |X|^2\right)\right) v \\
&\qquad + \left(\frac{A_s}{A} - \frac{L_s}{L} - B\frac{d}{2}\right) iv + \left(-\frac{L_s}{L} - B\right) i\Lambda v + i\left(2L\beta - \frac{X_s}{L}\right)\cdot\nabla v \\
&\qquad + \left(-2L^3 X + LB\beta - L\beta_s - \frac{B}{2}\frac{X_s}{L}\right)\cdot yv \\
&\qquad + \Delta_y v + \left[\frac{B_s}{4} - \left(\frac{B^2}{4} + L^4\right) - \frac{B}{2}\frac{L_s}{L}\right]|y|^2 v \Big\}(s,y),
\end{aligned}$$

where $\Lambda v := y\cdot\nabla v$.

Now choose wisely the parameters and their derivatives. . .

Plug previous ansatz in the Harmonic Oscillator (omit j subscript):

$$\begin{aligned} i\partial_t u + \Delta_x u - |x|^2 u \\ = \frac{A}{L^3} e^{i\gamma + iL\beta\cdot y - i\frac{B}{4}|y|^2} \Big\{ i\partial_s v + \left(-\gamma_s + \beta\cdot X_s - L^2\left(|\beta|^2 + |X|^2\right)\right) v \\ + \left(\frac{A_s}{A} - \frac{L_s}{L} - B\frac{d}{2}\right) iv + \left(-\frac{L_s}{L} - B\right) i\Lambda v + i\left(2L\beta - \frac{X_s}{L}\right)\cdot\nabla v \\ + \left(-2L^3 X + LB\beta - L\beta_s - \frac{B}{2}\frac{X_s}{L}\right)\cdot yv \\ + \Delta_y v + \left[\frac{B_s}{4} - \left(\frac{B^2}{4} + L^4\right) - \frac{B}{2}\frac{L_s}{L}\right]|y|^2 v \Big\}(s,y). \end{aligned}$$

where $\Lambda v := y\cdot\nabla v$.

Now choose wisely the parameters and their derivatives...

[red box] = 0, [blue box] = -1.

If the choice of parameters is valid/possible, it only remains:

$$(i\partial_t u + \Delta_x u - |x|^2 u)(t,x) = \frac{A}{L^3} e^{i\gamma + iL\beta\cdot y - i\frac{B}{4}|y|^2} \left\{ i\partial_s v + \Delta_y v - |y|^2 v \right\}(s,y).$$

For this choice of parameters,

$$(i\partial_t + \Delta_x - |x|^2) u(t,x) = 0 \quad \Longleftrightarrow \quad (i\partial_s + \Delta_y - |y|^2) v(s,y) = 0. \quad (2.1)$$

v satisfying RHS of (2.1) can be decomposed in Hermite basis:

$$\left\{ \varphi_n := H_{n_1} \cdots H_{n_d} : n \in \mathbb{N}^d \right\},$$

thus[3]

$$v(0,y) = \sum_{n\in\mathbb{N}^d} v_n \varphi_n(y) \implies v(s,y) = \sum_{n\in\mathbb{N}^d} v_n e^{-(2|n|+d)is} \varphi_n(y)$$

[3] because $(-\Delta_y + |y|^2)\varphi_n = (2|n| + d)\varphi_n$

Parameters

Parameters chosen so that:

$$
\left|
\begin{aligned}
&\gamma_s - \beta \cdot X_s + L^2 \left(|\beta|^2 + |X|^2 \right) = 0 \\
&\frac{A_s}{A} - \frac{L_s}{L} - \frac{B}{2} d = 0 \\
&- \frac{L_s}{L} - B = 0 \\
&2L\beta - \frac{X_s}{L} = 0 \\
&- 2L^3 X + LB\beta - L\beta_s - \frac{BX_s}{2L} = 0 \\
&\frac{B_s}{4} - \left(\frac{B^2}{4} + L^4 \right) - \frac{B}{2} \frac{L_s}{L} = -1.
\end{aligned}
\right.
\tag{2.2}
$$

Explicit expressions:

$$A(t) = A(0)\left(\frac{L(t)}{L(0)}\right)^{\frac{2-d}{2}},$$
$$L(t)^2 = 2h(t) - \cos(\xi(t))\sqrt{4h(t)^2-1},$$
$$B(t) = 2\sin(\xi(t))\sqrt{4h(t)^2-1},$$
$$X_i(t) = \sin(\theta_i(t))\sqrt{2a_i(t)}, \quad i=1,\dots,d,$$
$$\beta_i(t) = \cos(\theta_i(t))\sqrt{2a_i(t)}, \quad i=1,\dots,d,$$
$$\gamma(t) = \gamma(0) + \sum_{l=1}^{d}\frac{a_l(0)}{2}\left[\sin(2\theta_l(t)) - \sin(2\theta_l(0))\right]$$
$$s(t) = -\frac{1}{2}\arctan\left(\left(2h(0)+\sqrt{4h(0)^2-1}\right)\tan\left(\frac{\xi(0)}{2}-2t\right)\right) + \frac{1}{2}\arctan\left(\left(2h(0)+\sqrt{4h(0)^2-1}\right)\tan\left(\frac{\xi(0)}{2}\right)\right) + m_t\frac{\pi}{2},$$

$$a(t) = a(0), \quad \theta(t) = \theta(0)+2t, \quad h(t) = h(0), \quad \xi(t) = \xi(0)-4t.$$

where, if $m_0 \in \mathbb{Z}$ is such that $\frac{\xi(0)}{2} \in m_0\pi + \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $m_t \in \mathbb{Z}$ is defined by $\frac{\xi(t)}{2} \in (m_0 - m_t)\pi + \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$u(t,x) := \frac{A_j}{L_j} e^{i\gamma_j + iL_j\beta_j \cdot y_j - i\frac{B_j}{4}|y_j|^2} v(s_j, y_j),$$

Explicit integrations of parameters

Hermite decomposition + time integration

Content

We only need to solve (1.1)

$$i\partial_t \psi = |\psi|^2 \psi,$$

thanks to time-splitting.

Idea: need the bubble decomposition for the linear part, so Dirac-Frenkel principle for the nonlinear part.

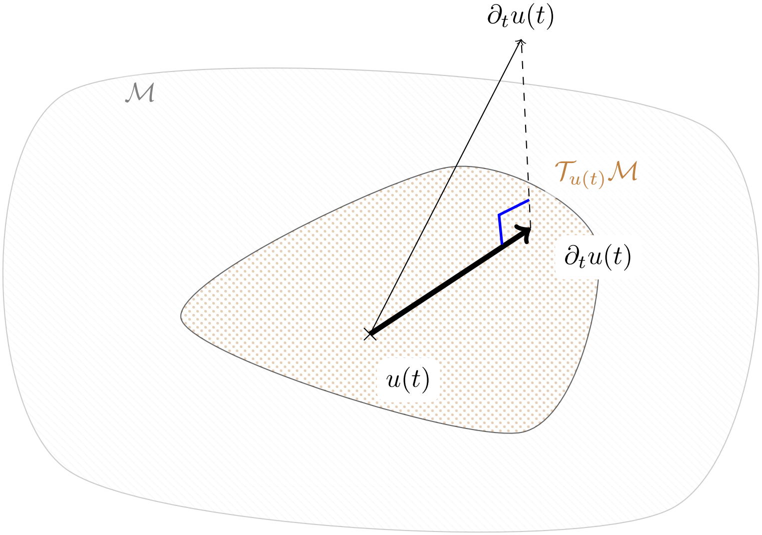
Dirac-Frenkel principle

Orthogonal projection of $|\psi|^2\psi$ onto the manifold corresponding to a bubble decomposition.

$$\mathcal{M} := \left\{ u \in \mathbb{L}^2(\mathbb{R}^d) \middle| \begin{array}{c} u(x) = \sum_{j=1}^{N} \frac{A_j}{L_j} e^{i\gamma_j + i\beta_j\cdot(x-X_j) - \frac{2+iB_j}{4L_j^2}|x-X_j|^2}, \\ A_j, B_j, \gamma_j \in \mathbb{R},\, L_j \in \mathbb{R}_+^*,\, X_j, \beta_j \in \mathbb{R}^d \end{array} \right\}.$$

Goal:

$$\text{Find } \partial_t u(t) \in \mathcal{T}_{u(t)}\mathcal{M}, \quad \text{such that}$$
$$\langle f, i\partial_t u(t)\rangle = \langle f, u(t)|u(t)|^2\rangle,\ \forall f \in B_{u(t)}.$$


$\partial_t u(t)$
$\mathcal{M}$
$\mathcal{T}_{u(t)}\mathcal{M}$
$\partial_t u(t)$
$u(t)$

$$\text{Find } \partial_t u(t) \in \mathcal{T}_{u(t)}\mathcal{M}, \quad \text{such that}$$
$$\langle f, i\partial_t u(t)\rangle = \langle f, u(t)|u(t)|^2\rangle, \forall f \in B_{u(t)}$$
$$\implies \mathbf{A}\mathbf{E} = \mathbf{S}.$$

A: Gram matrix of the projection onto $\mathcal{T}_{u(t)}\mathcal{M}$, computed *analytically*.
S: inner products between $u(t)|u(t)|^2$ and $B_{u(t)}$, computed *analytically*.

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Test case 1:

$$\psi(t=0,x) = \pi e^{-\frac{|x-\mu_1|^2}{2}} + 2e^{-\frac{|x-\mu_2|^2}{2}}, \quad x \in \mathbb{R}^2, \quad \mu_1 = (0,2),\ \mu_2 = (1,0).$$

Test case 2:

$$\psi(t=0,x) = e^{-|x-\mu_3|^2} e^{i \cosh |x-\mu_3|}, \quad x \in \mathbb{R}^2, \quad \mu_3 = (1,1).$$

Test case 2:

$$\psi(t=0,x) = \begin{cases} \sqrt{M^2 - |x|^2} e^{i \cosh \sqrt{x_1^2 + x_2^2}}, & |x|^2 < M^2 \\ 0 & \text{otherwise} \end{cases}, \quad M = 4.$$

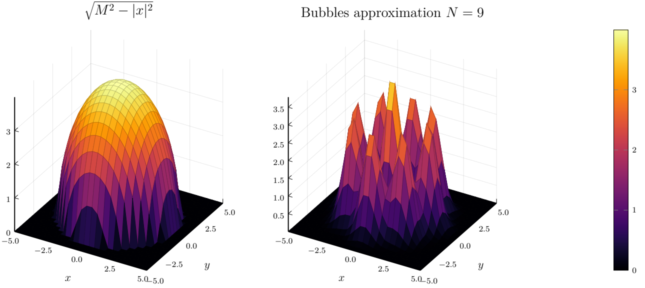
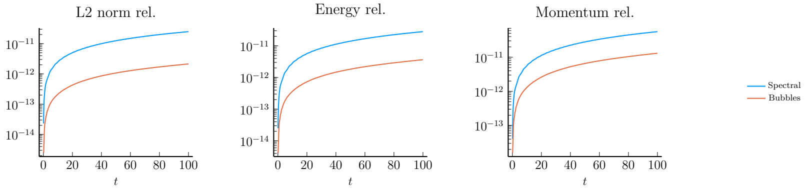
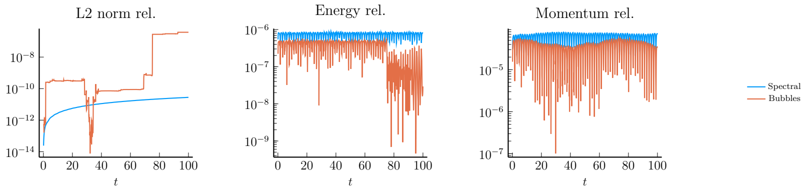


Figure: Approximation of $x \mapsto \sqrt{M^2 - |x|^2}$ as a sum of bubbles

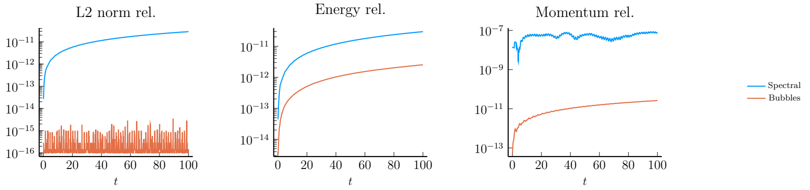


(a) (HO)

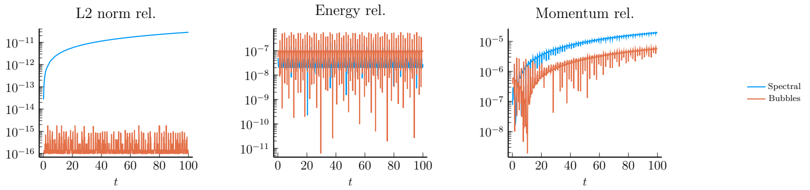


(b) (cNLS)

Figure: Test case 1, $\Delta t = 10^{-3}$

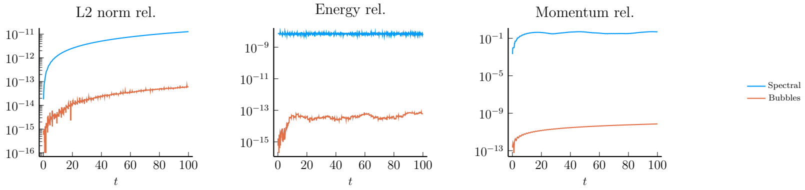


(a) (HO)

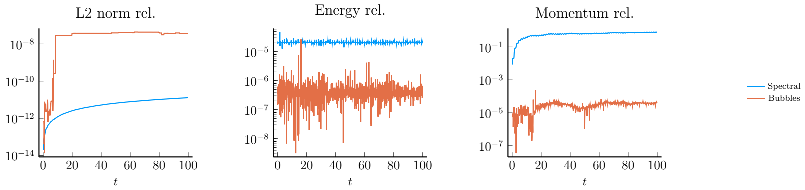


(b) (cNLS)

Figure: Test case 2, $\Delta t = 10^{-3}$



(a) (HO)



(b) (cNLS)

Figure: Test case 3, $\Delta t = 10^{-3}$

Thank you!