The spectral concentration problem

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Motivation

Real-life signals or measurements can be assimilated to compactly supported functions in Fourier and space.

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Real-life signals or measurements can be assimilated to compactly supported functions in Fourier and space.

Examples:

- "… AHHHHHHHHHHHHH …": time-limited sound, with a finite range of frequencies (human voice);
- More generally, signals coming from natural processes (geomagnetism, geophysics, biomedical, planetary sciences, …) are spatially and spectrally localized;
- Fourier optics (long range propagation $\approx$ Fourier transform).

Fourier notations:

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x} dx,$$

$$\mathcal{F}^{-1}[g](x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} g(\xi) e^{i\xi\cdot x} d\xi.$$

Space-limiting operator

Let $T > 0$, and $\mathfrak{D} := \left\{ f \in \mathbb{L}^2(\mathbb{R}) \middle| \text{supp } f \subset \mathbf{1}_{[-T,T]} \right\}$. A function $f \in \mathfrak{D}$ is said to be *space-limited*.

Define $\mathcal{D}$ the *space-limiting* operator: for $f \in \mathbb{L}^2(\mathbb{R})$,

$$(\mathcal{D}f)(x) := f(x)\mathbf{1}_{[-T,T]}(x), \quad x \in \mathbb{R}.$$

Band-limiting operator

Let $\Omega > 0$, and $\mathfrak{B} := \left\{ f \in \mathbb{L}^2(\mathbb{R}) \middle| \mathrm{supp}\ \hat{f} \subset \mathbf{1}_{[-\Omega,\Omega]} \right\}$. That is,

$$f \in \mathfrak{B} \implies f(x) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\xi) e^{i\xi \cdot x} d\xi.$$

A function $f \in \mathfrak{B}$ is said to be *band-limited*.

Define $\mathcal{B}$ the *band-limiting operator*: for $f \in \mathbb{L}^2(\mathbb{R})$,

$$(\mathcal{B}f)(x) := \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\xi) e^{i\xi \cdot x} d\xi.$$

Goal

Find functions that are the most "concentrated" in both space and Fourier domains.

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Define the *concentration ratio*

$$\lambda := \frac{\|\mathcal{B}\mathcal{D}f\|^2_{\mathbb{L}^2(\mathbb{R})}}{\|f\|^2_{\mathbb{L}^2(\mathbb{R})}} \in (0,1).$$

We look for

$$\arg\max_{f\in\mathbb{L}^2(\mathbb{R})} \lambda.$$

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$$\arg \max_{f \in \mathbb{L}^2(\mathbb{R})} \lambda.$$

Intuitively: such a maximizer f is very slightly modified when the operator $\mathcal{B}\mathcal{D}$ is applied, i.e. $\mathcal{D}f \approx f$ and $\mathcal{B}f \approx f$.

sinc kernel

We compute: for $f \in \mathbb{L}^2(\mathbb{R})$,

$$\lambda = \frac{\|\mathcal{B}\mathcal{D}f\|^2_{\mathbb{L}^2(\mathbb{R})}}{\|f\|^2_{\mathbb{L}^2(\mathbb{R})}} = \frac{\int_{-T}^{T}\int_{-T}^{T}\frac{\sin(\Omega(t-s))}{\pi(t-s)}f(t)\overline{f(s)}dtds}{\int_{-T}^{T}|f(t)|^2dt}.$$

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Maximizer of λ is given by $f \in \mathbb{L}^2(\mathbb{R})$ eigenfunction of an integral operator associated to largest eigenvalue $\lambda_1 \in (0, 1)$.

Integral operator

The integral operator at hand is defined by

$$(\tilde{\mathcal{K}} f)(\tilde{x}) := \int_{-T}^{T} \frac{\sin\left(\Omega(\tilde{y}-\tilde{x})\right)}{\pi(\tilde{y}-\tilde{x})} f(\tilde{y}) d\tilde{y}, \quad \tilde{x} \in [-T, T].$$

After a space renormalization, we can consider the *concentration operator*

Concentration operator

$$\left(\mathcal{K} f\right)(x) = \int_{-1}^{1} \frac{\sin\left(\Omega T(y-x)\right)}{\pi(y-x)} f(y) dy, \quad x \in [-1, 1].$$

Denote $c := \Omega T$.

We are looking for its eigenpairs $(\boldsymbol{\lambda}_{\mathbf{j}}, \boldsymbol{\psi}_{\mathbf{j}})$.
NB: we can extend ψ_j to $\mathbb{R}$.

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Some properties of the concentration operator

Proposition

- The eigenvalues are real and countable:

$$1 > \lambda_0 > \lambda_1 > \cdots > 0,$$

  and the eigenfunctions $\{\psi_j\}_{j\in\mathbb{N}}$ are real, and either even or odd.
- They are complete in $\mathbb{L}^2([-1,1])$ as well as $\mathbb{L}^2(\mathbb{R})$:

$$\int_{\mathbb{R}} \psi_i(x)\psi_j(x)dx = \delta_{i,j}, \quad \int_{-1}^{1} \psi_i(x)\psi_j(x)dx = \lambda_i\delta_{i,j}.$$

- **Commuting property**...

Commuting property

Slepian and Pollak showed[1] that there exists $\mathcal{P}$ a differential operator such that $\mathcal{K}\mathcal{P} = \mathcal{P}\mathcal{K}$, and

Commuting differential operator

$$(\mathcal{P}f)(x) = \frac{d}{dx}\left[(1 - x^2)\frac{df}{dx}(x)\right] - cx^2 f(x).$$

This commutation property is at the heart of most papers, and can be used to obtain efficiently the eigenvectors ψ_j of $\mathcal{K}$.

[1] D. Slepian and H. O. Pollak. "Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty - I". In: *Bell System Technical Journal* 40.1 (Jan. 1961).

A fool's attempt

Why bother with the commuting differential operator??

Let's try a classical eigendecomposition algorithm...

Discretize $\mathcal{K}$ using N discretization points.

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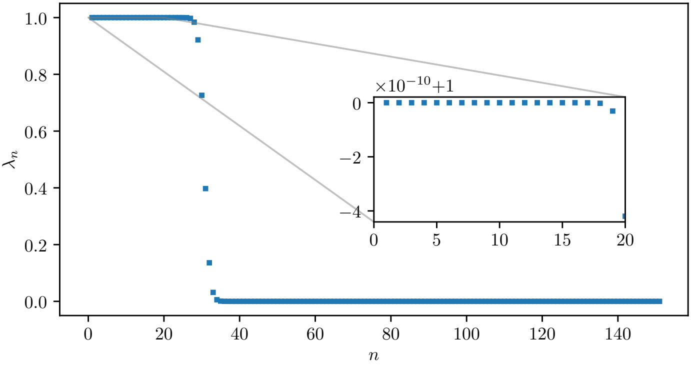

Figure: Eigenvalues of the discretized concentration matrix,
$N = 151, \Omega = 0.1 \cdot 2\pi$.

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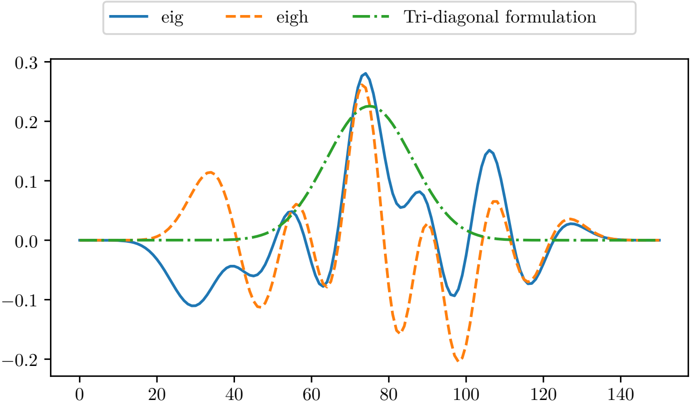

Figure: Eigenvectors of the discretized concentration matrix, $N = 151, \Omega = 0.1 \cdot 2\pi$.

Why the numerical issue?

Numerically, we are in the following situation:

Lemma

Let $\mathbf{A}$ a $n \times n$ matrix, with an eigenvalue λ of multiplicity $m \leq n$. Let $u_1, \dots, u_m$, m independant eigenvectors of $\mathbf{A}$ associated to the eigenvalue λ. Then any linear combination of $u_1, \dots, u_m$ is also an eigenvector of $\mathbf{A}$ associated to λ.

Proof.

Let $c_1, \dots, c_m \in \mathbb{C}$,

$$\mathbf{A}\left(\sum_{i=1}^{m} c_i u_i\right) = \sum_{i=1}^{m} c_i \mathbf{A} u_i = \sum_{i=1}^{m} c_i \lambda u_i = \lambda \left(\sum_{i=1}^{m} c_i u_i\right).$$

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Why consider a generalized problem?

Motivation: again, Fourier optics.

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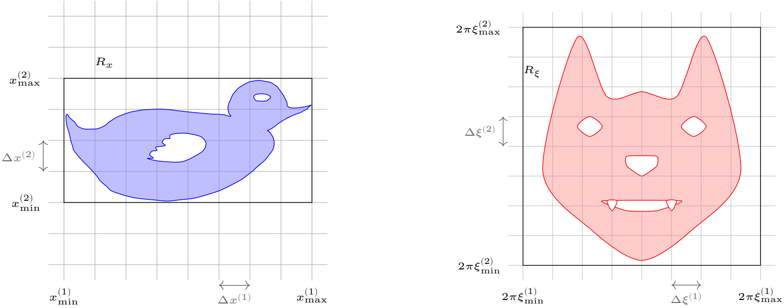

Figure: duck: Hugo E.©

Generalized masks

Introduce the space and Fourier filters/masks: $m_S, \widehat{m_F} \in \mathbb{L}^2(\mathbb{R}^d; \mathbb{C})$.

Similarly to the introductory example, consider the space- and Fourier-limiting operators:

$$(\mathcal{M}_S g)(x) := m_S(x) g(x), \qquad (\mathcal{M}_F g)(x) := \mathcal{F}^{-1}\left[\widehat{m_F}\mathcal{F}[g]\right](x).$$

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Introductory example

$$(\mathcal{D}f)(x) := f(x)\mathbf{1}_{[-T,T]}(x), \qquad (\mathcal{B}f)(x) := \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \hat{f}(\xi) e^{i\xi\cdot x} d\xi,$$

so

$$m_S(x) = \mathbf{1}_{[-T,T]}, \qquad \widehat{m_F}(\xi) = \mathbf{1}_{[-\Omega,\Omega]}(\xi).$$

Goal

Consider the maximization problem:

$$\arg\max_{f\in\mathbb{L}^2(\mathbb{R}^d;\mathbb{C})} \frac{\|\mathcal{M}_F\mathcal{M}_S f\|^2_{\mathbb{L}^2(\mathbb{R}^d;\mathbb{C})}}{\|f\|^2_{\mathbb{L}^2(\mathbb{R}^d;\mathbb{C})}}, \tag{1}$$

its solution is again given by the eigenfunctions of some integral operator...

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Generalized concentration kernel

For $f \in \mathbb{L}^2(\mathbb{R}^d;\mathbb{C})$, define

$$(\mathcal{K}f)(x) := \int_{\mathbb{R}^d} K(x,y) f(y) dy, \quad x \in \mathbb{R}^d,$$

where

$$K(x,y) = m_S(x)\overline{m_S(y)}\mathcal{F}^{-1}\left[|\widehat{m_F}|^2\right](y-x), \quad x,y \in \mathbb{R}^d.$$

Proposition

The concentration operator $\mathcal{K}$ is a *Hilbert-Schmidt* operator and:

1. The kernel K is Hermitian, and the operator $\mathcal{K}$ is self-adjoint, compact, and positive semi-definite.
2. The countable family $\{\psi_i\}_{i=1}^{\infty}$ of eigenfunctions of $\mathcal{K}$ is orthonormal for the usual $\mathbb{L}^2(\mathbb{R}^d;\mathbb{C})$ inner product, the associated eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ are real, nonnegative, and we can order them so that $1 > \lambda_i \geq \lambda_{i+1} \geq 0$, $i \geq 1$.
3. The orthonormal basis of eigenfunctions $\{\psi_i\}_{i=1}^{\infty}$ solve the maximization problem (1), and the maximal values attained are the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$.
4. For large n, $\lambda_n = o(n^{-1/2})$.
5. Suppose $|\widehat{m_F}|^2$ is even, and m_S is real, then $\mathcal{K}$ is real-valued for real inputs.

Some experiments...

The generalized problem accepts any $\mathbb{L}^2$ function for m_S and $\widehat{m_F}$, we now focus on 2d examples with

$$m_S = \mathbf{1}_{D_1}, \quad \text{and} \quad \widehat{m_F} = \mathbf{1}_{B(0,0.3\times 2\pi)},$$

with $D_1 \subset \mathbb{R}^2$.

The generalized concentration operator is discretized using $N_1 \times N_2$ points $\rightarrow$ matrix $\mathbf{K}$.

A fool's attempt (again)

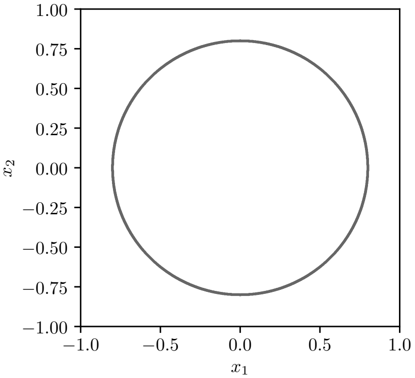

Figure: $m_S = \mathbf{1}_{B(0,0.8)}$.

A fool's attempt (again)

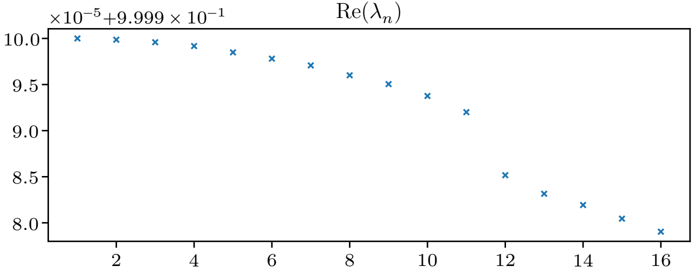

Figure: Eigenvalues with a direct decomposition, in the case $D_1 = \text{Disc}(0, 0.8)$. They are the exact eigenvalues up to some tolerance $\eta = 10^{-5}$. $N_1 = 50, N_2 = 50$.

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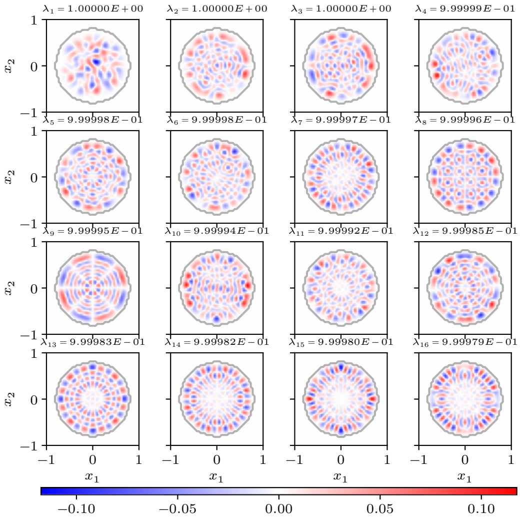

Figure: Eigenvectors obtained with a direct decomposition, in the case $D_1 = \text{Disc}(0, 0.8)$. $N_1 = 50, N_2 = 50$.

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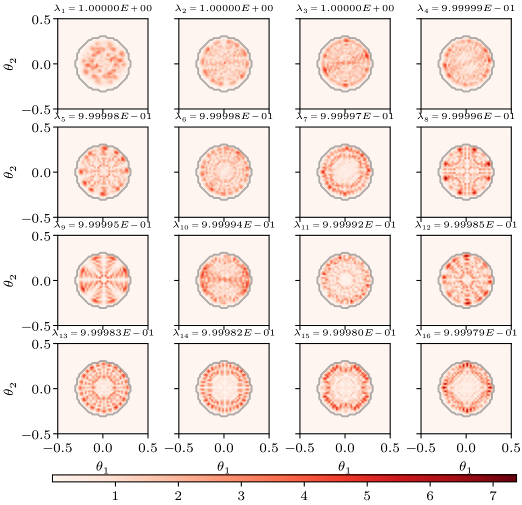

Figure: Fourier transform of eigenvectors obtained with a direct decomposition, in the case $D_1 = \text{Disc}(0, 0.8)$. $N_1 = 50, N_2 = 50$.

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Back to 1D

The crucial point is that the eigenvalues depends on $c = \Omega T$:

$$T \to 0 \implies c \to 0 \implies \lambda_j \to 0, \forall j.$$

Consider a scaling

$$\begin{aligned}\mathbf{1}_{[-\mu(\varepsilon),\mu(\varepsilon)]} &\underset{\varepsilon \to 0}{\longrightarrow} \mathbf{1}_{[-1,1]} \\ &\underset{\varepsilon \to \infty}{\longrightarrow} \mathbf{1}_{\{0\}}.\end{aligned}$$

Q.: What happens to eigenvalues λ_n when ε varies?

Back to 1D

The only requirement is that

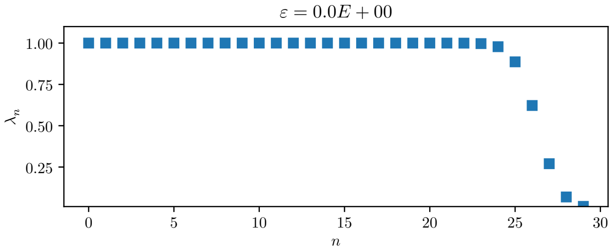
$$\mu(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 1, \quad \text{and} \quad \mu(\varepsilon) \underset{\varepsilon\to +\infty}{\longrightarrow} 0,$$

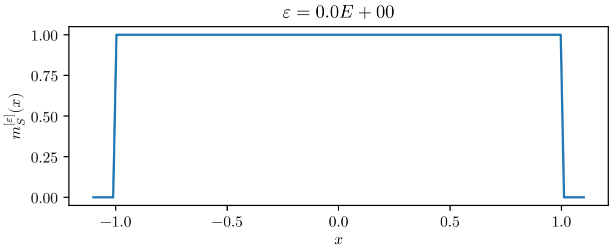
so consider for example

$$\mu(\varepsilon) = \frac{1}{(1+\varepsilon^4)^{1/4}}.$$

Modified space mask: $m_S^{[\varepsilon]} := \mathbf{1}_{[-\mu(\varepsilon),\mu(\varepsilon)]} \to \mathscr{K}^{[\varepsilon]} \to \mathbf{K}^{[\varepsilon]}$.

Behavior of eigenvalues





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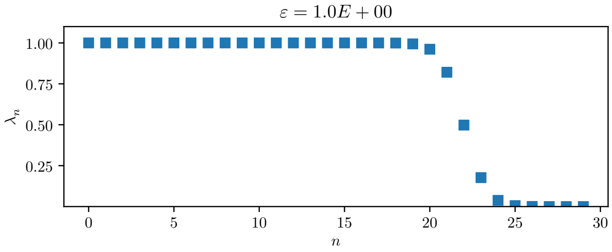

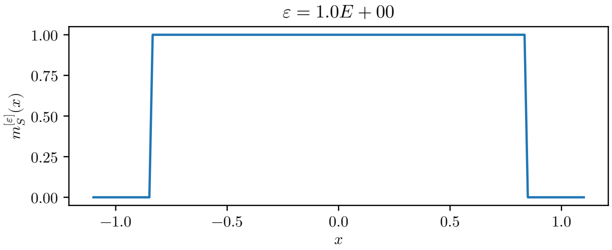

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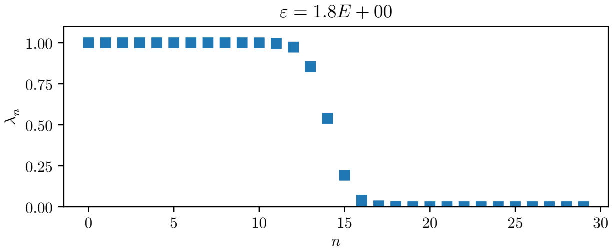

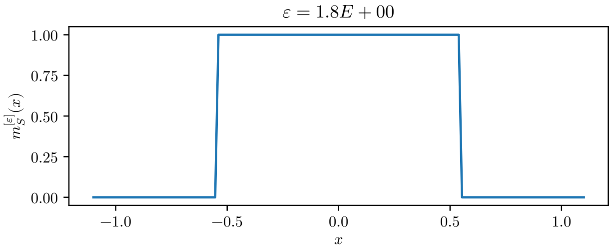

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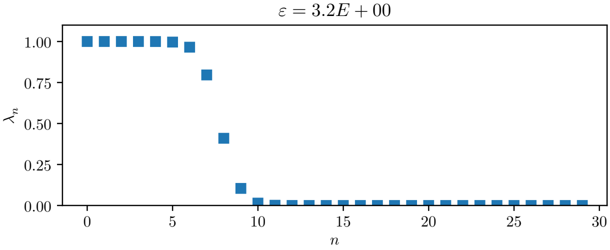

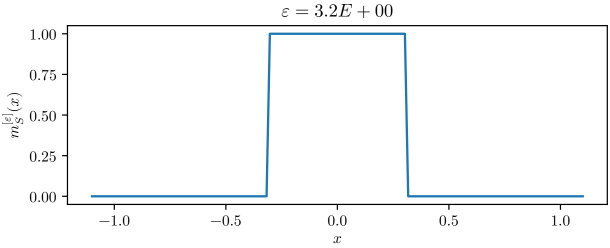

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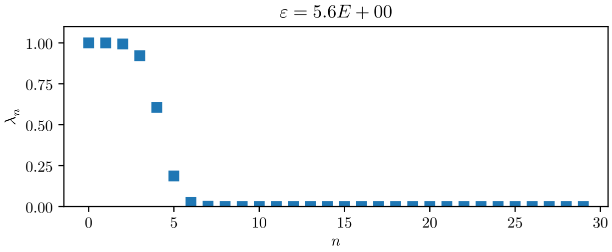

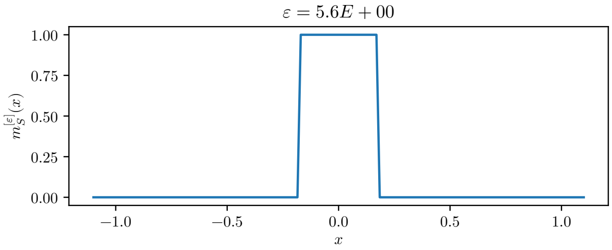

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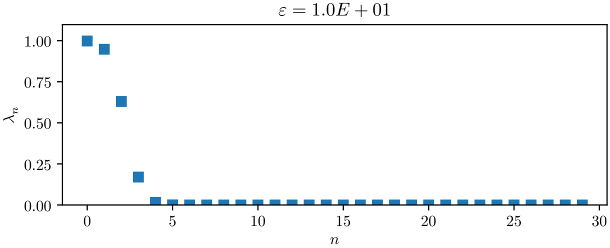

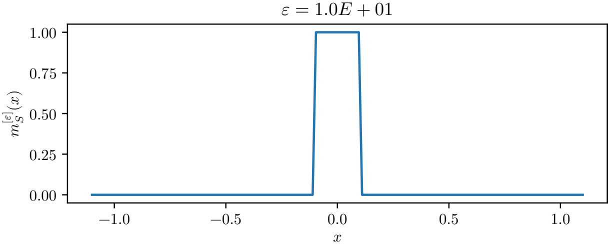

What's the point?

When $c \to 0$, the eigenvalues move away from each other.

$\downarrow$

One can get the associated eigenvectors without confusion.

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Algorithm idea:

1. start with $\varepsilon \gg 1 \implies \lambda_j^{[\varepsilon]} \gg \lambda_{j+1}^{[\varepsilon]} \implies v_j^{[\varepsilon]}$ e.$\vec{\text{v}}$. of $\mathbf{K}^{[\varepsilon]}$ OK;
2. compute $\nu := v_j^{[\varepsilon]} \mathbf{K}^{[0]} v_j^{[\varepsilon]}$ (concentration ratio of $v_j^{[\varepsilon]}$);
3. if ν close enough to $\lambda_j^{[0]}$, save $v_j^{[\varepsilon]}$ approx. of $v_j^{[0]}$;
4. take ε smaller and repeat.

Not truly eigenvectors of the initial problem, but...

- The eigenvectors of the modified problem are "close enough" to eigenvectors of the initial concentration problem;
- They are still an orthonormal basis of $\mathbb{R}^{N_1 \cdots N_d}$.

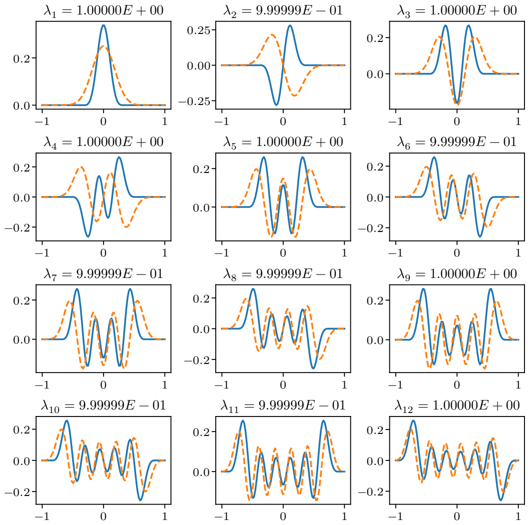
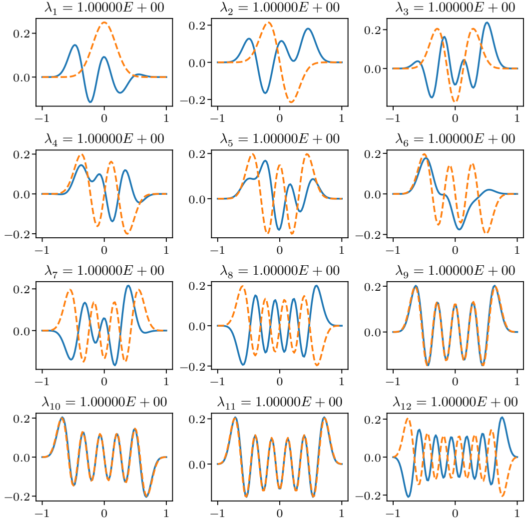


Figure: First 12 eigenpairs with the varying spacemask procedure (solid blue curve), in 1D, with $N = 100$, $\Omega = 0.1 \cdot 2\pi$. The exact eigenvectors are given by orange dash curves.

To be compared with a direct eigendecomposition...



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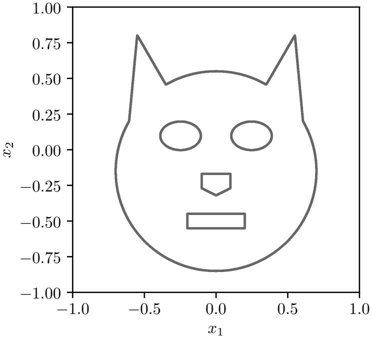
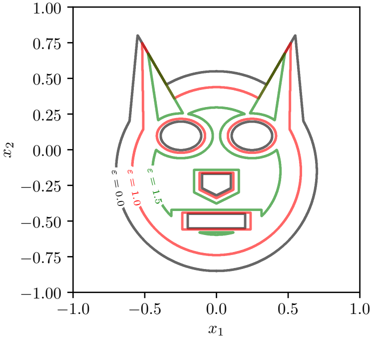
A 2D example

Same idea works in dimension $d \geq 1$.



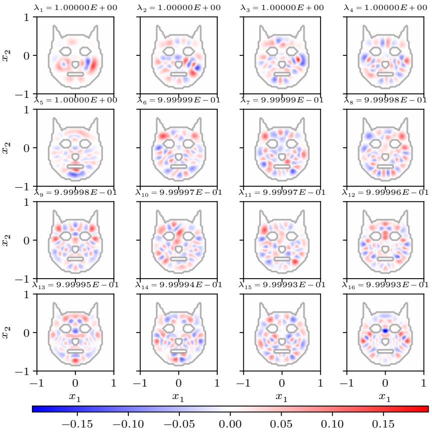
(a) Cat-head shape.



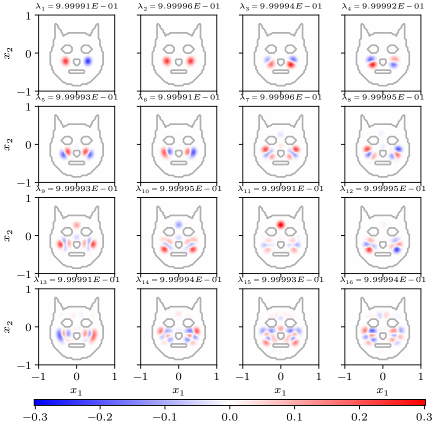
(b) Set-valued function $D_1(\varepsilon)$, with $D_1 = D_1(0) =$ Cat-head.

Figure: A (poorly drawn) cat-head shape, as well as the set-valued function $D_1(\varepsilon)$, decreasing for the inclusion relation, and such that $D_1(0) =$ Cat-head.

Cat-head

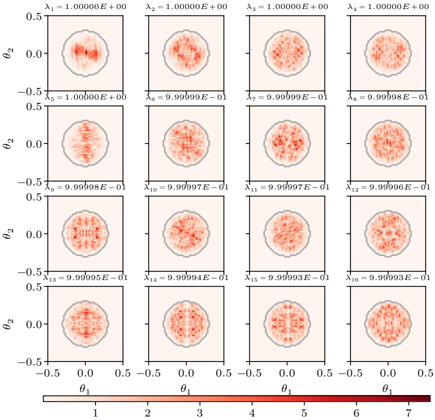


(a) Direct eigendecomposition.



(b) Varying space mask procedure.

Cat-head in Fourier



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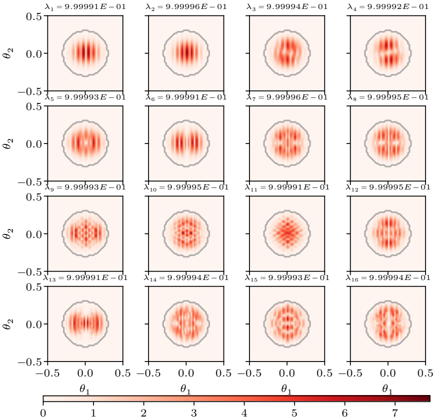

(b) Varying space mask procedure.

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Conclusion & Perspectives

Conclusions:

- We can recover almost-maximizers of the concentration ratio, down to some given numerical tolerance;
- We can bypass the issue of eigenvalues being too close to each other;
- The approximate eigenvectors are still a basis of $\mathbb{L}^2(\mathbb{R})$, and do not depend on the eigenvector algorithm used $\rightarrow$ numerically more robust.

Approximate eigenvectors obtained with the varying space mask procedure are a good alternative to the true eigenvectors!

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Approximate eigenvectors obtained with the varying space mask procedure are a good alternative to the true eigenvectors!

Perspectives:

- Better understand why the varying space mask procedure yields vectors more localized than those we are looking for;
- Try the dynamical low-rank approach with a smoothing of the spacemask, to avoid "singularities" when eigenvalues get too close to each other;
- Try extrapolation techniques to really go down to $\varepsilon \approx 0$.


The end.