Composite Finite Volume schemes and Source Term discretization.

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Overview

1. Finite Volume flux schemes: homogeneous case
   - Edge schemes
   - Node schemes
   - Composite schemes
2. Source Term discretization ($\theta = 1$)
   - Naive discretization
   - Enhanced consistency
   - Numerical Results

Framework

2D Euler equations, with gravity: $x \in \Omega \subset \mathbb{R}^2$, $t \in \mathbb{R}^+$,

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{U}) = 0, \\ \partial_t(\rho \mathbf{U}) + \operatorname{div}(\rho \mathbf{U} \otimes \mathbf{U} + P I_2) = -\rho \mathbf{g}, \\ \partial_t(\rho E) + \operatorname{div}(\rho E \mathbf{U} + P \mathbf{U}) = -\rho \mathbf{g} \cdot \mathbf{U}, \end{cases} \tag{1}$$

$$P = (\gamma - 1)\rho e, \quad e = E - \frac{1}{2} |\mathbf{U}|^2,$$

$$\mathbf{U} = (u_1, u_2) \in \mathbb{R}^2, \quad \mathbf{g} = (g_1, g_2) \in \mathbb{R}^2.$$

Conservative form of (1)

Letting

$$\mathcal{U} := \begin{pmatrix} \rho \\ \rho \mathbf{U} \\ \rho E \end{pmatrix} \in \mathbb{R}^4,$$

we have

$$\partial_t \mathcal{U} + \operatorname{div} \mathcal{F}(\mathcal{U}) = \mathbf{S}, \tag{2}$$

where

$$\mathcal{F}(\mathcal{U}) := \begin{pmatrix} \rho u_1 & \rho u_2 \\ \rho u_1^2 + P & \rho u_1 u_2 \\ \rho u_1 u_2 & \rho u_2^2 + P \\ (\rho E + P) u_1 & (\rho E + P) u_2 \end{pmatrix}, \quad \mathbf{S}(\mathcal{U}) = \begin{pmatrix} 0 \\ -\rho g_1 \\ -\rho g_2 \\ -\mathbf{g} \cdot (\rho \mathbf{U}) \end{pmatrix}.$$

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Finite Volume Method: homogeneous case

We consider the following conservative system to be solved on $\Omega \subset \mathbb{R}^2$:

$$\partial_t \mathcal{U} + \operatorname{div} \mathcal{F}(\mathcal{U}) = 0, \tag{3}$$

where $\mathcal{U}(t,x) \in \mathbb{R}^4$ is the conservative unknown and $\mathcal{F} \in \mathbb{R}^{4,2}$ is the physical flux function.
We consider the following hypotheses:

- $\forall \xi \in \mathbb{R}^2$, with $|\xi| = 1$, the Jacobian matrix $J(\mathcal{U}, \xi) = \dfrac{\partial \mathcal{F}}{\partial \mathcal{U}} \cdot \xi$ is **diagonalizable**.
- The n eigenvalues $\lambda_i = 1, ..., n$ of J are real.
- Some additional technical assumptions.

Finite Volume Method

Finite volume method comes from integrating (3) on each cell Ω_j:

$$\int_{\Omega_j} \partial_t \mathcal{U} + \operatorname{div} \mathcal{F}(\mathcal{U}) = 0,$$

By applying the Green-Riemann formula, we get:

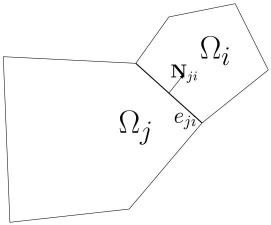
$$\partial_t \mathcal{U}_j(t) + \frac{1}{|\Omega_j|} \int_{\partial \Omega_j} \mathcal{F}(\mathcal{U}) \cdot N_j ds = 0,$$

where N_j is outward unit normal vector to Ω_j, and the discrete unknown are $\mathcal{U}_j(t) = \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathcal{U}(t,x)dx$. On each edge,

$$\int_{\partial \Omega_j \cap \partial \Omega_i} \mathcal{F}(\mathcal{U}) \cdot N_{ji} ds \approx |e_{ji}| \mathcal{G}(\mathcal{U}_j, \mathcal{U}_i) \cdot N_{ji}, \;\; e_{ji} = |\Omega_i \cap \Omega_j|. \tag{4}$$

Numerical flux $\mathcal{G}$ can be specified by any of the well-known schemes: **VFFC, Roe, Godunov, Rusanov, HLL**...

Requirements on the numerical flux $\mathcal{G}$



- **Local conservation:**

$$(\mathcal{G}(\mathcal{U}_j,\mathcal{U}_k) + \mathcal{G}(\mathcal{U}_k,\mathcal{U}_j)) \cdot N_j = 0,$$

- **Consistency:**

$$\mathcal{G}(\mathcal{U},\mathcal{U}) \cdot N_j = \mathcal{F}(\mathcal{U}) \cdot N_j,$$

Motivation

The pure edge finite volume schemes doesn't perform well in some cases, and may need a severe CFL constraint.

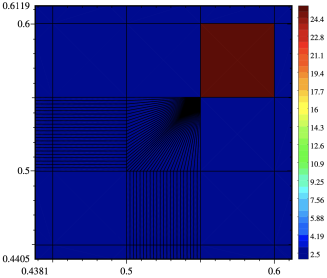
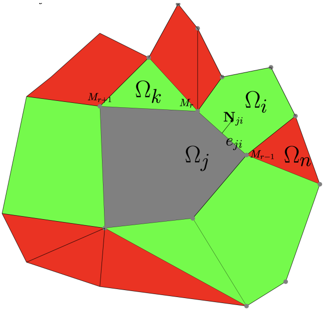


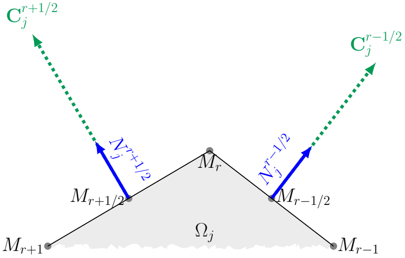
Figure 4: Numerical travel time: first order explicit finite volume pure edge scheme depends highly on cell/edge repartition. Consider a fluid initialy at rest and one cell (center $(0.575, 0.575)$) density field, the numerical arrival time in the cell at left/bottom corner (center $(0.475, 0.475)$) highly depends on the NUMBER of edges which separates them.

Figure: Example from [3]

Aim (1/2)

Extend the classical eulerian edge finite volume method to eulerian nodal finite volume method.





$$C_j^{r+1/2} := (M_r M_{r+1})^{\perp}$$

FV Edge scheme is given by (summing (4) over all edges of Ω_j):

$$|\Omega_j|\frac{\mathcal{U}_j^{n+1}-\mathcal{U}_j^n}{\Delta t}+\sum_{r+1/2\in\Omega_j}\mathcal{G}^{r+1/2}\cdot C_j^{r+1/2}=0, \qquad (5)$$

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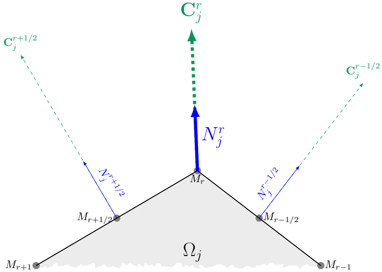
Let $C_j^r := \frac{1}{2}(M_{r-1}M_{r+1})^{\perp}$:



In (5), the sum is performed over degrees of freedom $dof \in \{r+1/2 \in \Omega_j\}$, one could also perform this sum over $dof \in \{r \in \Omega_j\}$, and obtain a *Node scheme*:

$$|\Omega_j|\frac{\mathcal{U}_j^{n+1}-\mathcal{U}_j^n}{\Delta t}+\sum_{r\in\Omega_j}\mathcal{G}^{r}\cdot C_j^{r}=0. \tag{6}$$

Aim (2/2)

Combine Edge Flux scheme (5) with Node Flux scheme (6) to obtain a Composite Flux scheme.

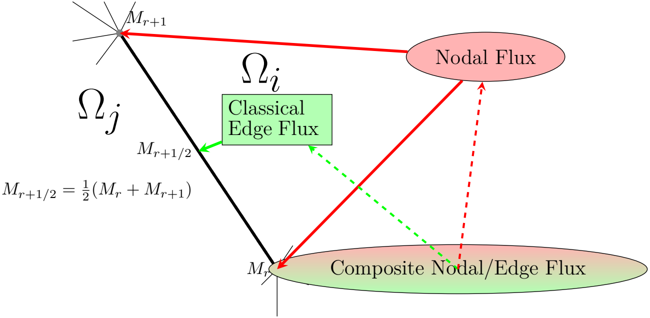


Figure: Composite Nodal/Edge fluxes: Edge fluxes involve only adjacent neighbor cells while nodal fluxes involve any cell sharing one of the end point edge.

$$\text{Composite} = \theta \text{Edge} + (1-\theta)\text{Node}, \quad \theta \in [0,1]. \tag{7}$$

Composite θ-scheme

A composite θ-scheme can be written as:

$$|\Omega_j|\frac{\mathcal{U}_j^{n+1}-\mathcal{U}_j^n}{\Delta t}+(1-\theta)\sum_{r\in\Omega_j}\mathcal{G}^r\cdot C_j^r+\theta\sum_{r+1/2\in\Omega_j}\mathcal{G}^{r+1/2}\cdot C_j^{r+1/2}=0. \quad (8)$$

The dependence on θ in (8), allows to easily recover the two well-known types of schemes:

$\theta = 0$ nodal scheme.

$\theta = 1$ edge scheme.

Improvements

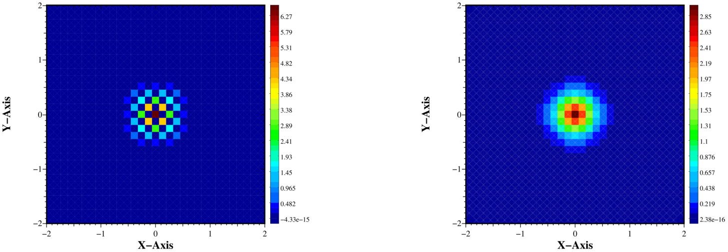


Figure 2: Numerical solution of hyperbolic P1 model on cartesian mesh with Dirac like Cauchy data see [11, 8]. Left: with a pure nodal polygonal scheme. Right: with a (composite) conical degenerate scheme. The pure nodal scheme exhibits some cross stencil unphysical phenomenom (here cured by the composite scheme, see Figure 5 and section below), both are first order in time and space.

Figure: Example from [3]

Overview

Naive discretization

When the system of conservation laws is completed with a source term.

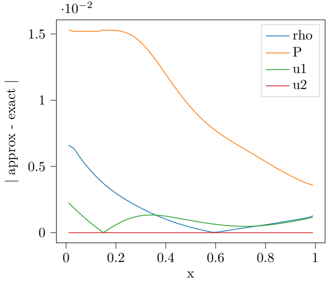
$$\mathcal{Q}_j = \frac{1}{|\Omega_j|} \int_{\Omega_j} S(x)dx \approx S(x_j),$$

approximate solution does not stay on exact stationary solutions, drifts away from it.

A steady-state for (1) in 1D:

$$\begin{cases} \rho(x) = e^{-gx} + 1, \\ \mathbf{U}(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ P(x) = C - gx + e^{-gx}, \end{cases}$$

where C is s.t. $P > 0$.



$N_x = 50, g = 5, \theta = 1, T_{final} = 0.1$

Enhanced Consistency

Objective: "Capture" the continuous stationary solutions (see [4, 2]).
Idea from [1]: works in 1D, disappointing results in 2D.

Let $\Phi : \Omega \mapsto \mathbb{R}^{4,2}$ such that

$$\operatorname{div} \Phi = S.$$

Criterion for "capturing" the continuous stationary solution:

$$\mathcal{F}(\mathcal{U}_j) = \Phi(x_j),\ \forall j.$$

The Euler equations (2) can be rewritten

$$\partial_t \mathcal{U} + \operatorname{div} \left[\mathcal{F}(\mathcal{U}) - \Phi\right] = 0.$$

BUT the numerical flux associated to

$$\partial_t \mathcal{U} + \operatorname{div} \tilde{\mathcal{F}}(\mathcal{U}) = 0$$

only involves $\tilde{\mathcal{F}}(\mathcal{U})$ at the center x_j of cells Ω_j.

Finding Φ

$$\operatorname{div} \Phi = \operatorname{div} \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \\ \Phi_{31} & \Phi_{32} \\ \Phi_{41} & \Phi_{42} \end{pmatrix} = \partial_x \begin{pmatrix} \Phi_{11} \\ \Phi_{21} \\ \Phi_{31} \\ \Phi_{41} \end{pmatrix} + \partial_y \begin{pmatrix} \Phi_{12} \\ \Phi_{22} \\ \Phi_{32} \\ \Phi_{42} \end{pmatrix} = S = \begin{pmatrix} 0 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix}.$$

Problem underconstrained, we need more information!

Zero-speed stationary solutions

We look at stationary solutions such that $\mathbf{U} = (0, 0)$. Then, solving (1) amounts to solving

$$\operatorname{div} \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} = \begin{pmatrix} S_2 \\ S_3 \end{pmatrix}.$$

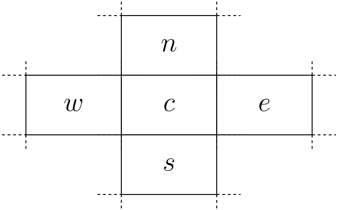
It makes sense to look for

$$\Phi = \begin{pmatrix} 0 & 0 \\ \phi & 0 \\ 0 & \phi \\ 0 & 0 \end{pmatrix} \quad \text{such that} \quad \begin{cases} \partial_x \phi = S_2 \\ \partial_y \phi = S_3 \end{cases}.$$

We consider $\mathbf{S}$ is polynomial of degree n and look for ϕ polynomial of degree $n + 1$.

Polynomial interpolation on Cartesian grid

$$\phi_j(x,y) = \alpha_j^1 x + \alpha_j^2 x^2 + \alpha_j^3 x^3 + \alpha_j^4 y + \alpha_j^5 y^2 + \alpha_j^6 y^3$$



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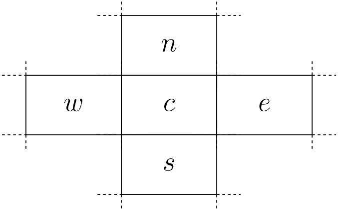

Interpolation is done by solving a linear system on each cell j:

$$\begin{pmatrix} 1 & 2x_j & 2x_j^2 & 0 & 0 & 0 \\ 1 & 2x_r & 2x_r^2 & 0 & 0 & 0 \\ 1 & 2x_l & 2x_l^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2y_j & 3y_j^2 \\ 0 & 0 & 0 & 1 & 2y_b & 3y_b^2 \\ 0 & 0 & 0 & 1 & 2y_t & 3y_t^2 \end{pmatrix} \begin{pmatrix} \alpha_j^1 \\ \alpha_j^2 \\ \alpha_j^3 \\ \alpha_j^4 \\ \alpha_j^5 \\ \alpha_j^6 \end{pmatrix} = \begin{pmatrix} S^1(x_j, y_j) \\ S^1(x_r, y_r) \\ S^1(x_l, y_l) \\ S^2(x_j, y_j) \\ S^2(x_b, y_b) \\ S^2(x_t, y_t) \end{pmatrix}$$

We consider a stationary solution of (1) given by

$$\begin{cases} \rho(t,x) = 1 \\ \mathbf{U}(t,x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ P(t,x) = -g_1 x_1 \end{cases} \quad \text{with} \quad \mathbf{S}(t,x) = \begin{pmatrix} 0 \\ -\rho g_1 \\ 0 \\ -\rho g_1 u_1 \end{pmatrix}$$

and where C s.t. $P > 0$. $g_1 = 0.1, \theta = 1, T_{final} = 0.1$.

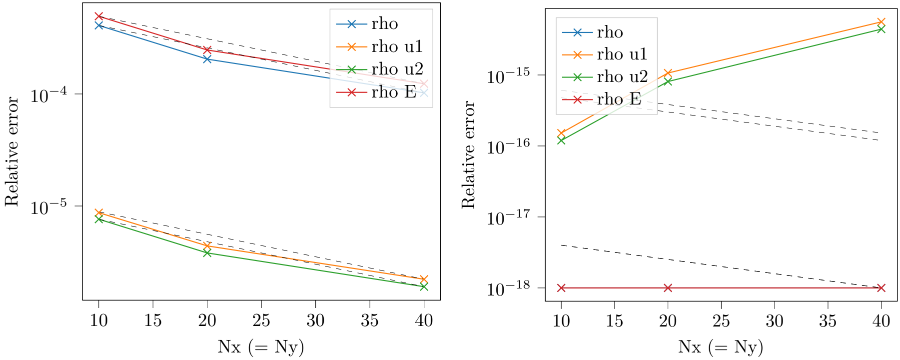

Figure: Left: naive discretization, right: polynomial reconstruction of Φ

We consider a stationary solution of (1) given by

$$\begin{cases} \rho(t,x) = 2x_1 + 1 \\ \mathbf{U}(t,x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ P(t,x) = -g_1 x_1^2 - g_1 x_1 + C \end{cases} \quad \text{with} \quad \mathbf{S}(t,x) = \begin{pmatrix} 0 \\ -\rho g_1 \\ 0 \\ -\rho g_1 u_1 \end{pmatrix}$$

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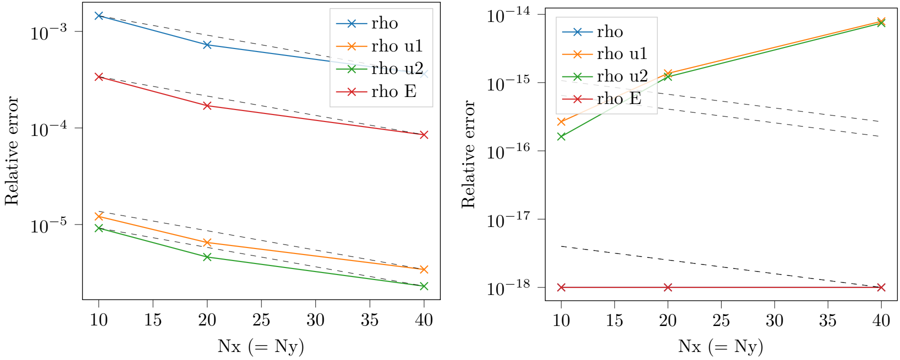

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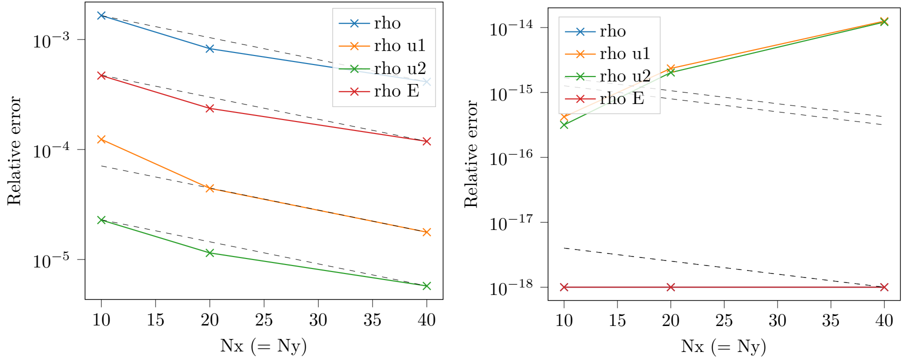
We consider a stationary solution of (1) given by

$$\begin{cases} \rho(t,x) = 3x_1^2 + 1 \\ \mathbf{U}(t,x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ P(t,x) = -g_1 x_1^3 - g_1 x_1 + C \end{cases} \quad \text{with} \quad \mathbf{S}(t,x) = \begin{pmatrix} 0 \\ -\rho g_1 \\ 0 \\ -\rho g_1 u_1 \end{pmatrix}$$

and where C s.t. $P > 0$. $g_1 = 0.1, \theta = 1, T_{final} = 0.1$.



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Perspectives

- Use node neighbors for interpolation
- Choose the degree of polynomial according to the number of neighbor cells
- Combine Enhanced Consistency with composite schemes and/or unstructured meshes
- If possible, adapt the method to non-zero speeds
- Consider other types of source terms (not only gravity!)
- Enhanced Consistency for viscosity schemes?

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That's all Folks!